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Riemann–Hilbert analysis and uniform convergence of rational interpolants to the exponential function☆

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Abstract

We study the asymptotic behavior of the polynomials p and q of degrees n, rational interpolants to the exponential function, defined by $p(z)e^{-z/2}+q(z)e^{z/2}=\mathcal{O}\,(\omega_{2n+1}(z))$, as z tends to the roots of ω_{2n+1}, a complex polynomial of degree $2n+1$. The roots of ω_{2n+1} may grow to infinity with n, but their modulus should remain uniformly bounded by $c\ \log(n)$, $c<1/2$, as $n\to\infty$. We follow an approach similar to the one in a recent work with Arno Kuijlaars and Walter Van Assche on Hermite–Padé approximants to e^z. The polynomials p and q are characterized by a Riemann–Hilbert problem for a 2×2 matrix valued function. The Deift–Zhou steepest descent method for Riemann–Hilbert problems is used to obtain strong uniform asymptotics for the scaled polynomials $p(2nz)$ and $q(2nz)$ in every domain in the complex plane. From these asymptotics, we deduce uniform convergence of general rational interpolants to the exponential function and a precise estimate on the error function. This extends previous results on rational interpolants to the exponential function known so far for real interpolation points and some cases of complex conjugate interpolation points.

Keywords: Riemann–Hilbert analysis; Rational interpolation; Uniform convergence; Exponential function

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1. Introduction

Let $B := B^{(n_1+n_2)} = \{z_i^{(n_1+n_2)}\}_{i=0}^{n_1+n_2}$, be a triangular sequence of complex interpolation points. We define the rational interpolants of type (n_1, n_2) to e^z such that

$$p_{n_1,n_2}(z) + q_{n_1,n_2}(z)e^z = \mathcal{O}\left(\omega_{n_1+n_2+1}(z)\right), \quad z \to z_i^{(n_1+n_2)}, \\ i = 0, \ldots, n_1+n_2, \tag{1.1}$$

where

$$\omega_{n_1+n_2+1}(z) = \prod_{i=0}^{n_1+n_2} \left(z - z_i^{(n_1+n_2)}\right)$$

and $\deg p_{n_1} \leqslant n_1$, $\deg q_{n_2} \leqslant n_2$. The polynomials p_{n_1,n_2} and q_{n_1,n_2} exist. Indeed, relation (1.1) is equivalent to a system of n_1+n_2+1 linear homogeneous equations for the n_1+n_2+2 unknown coefficients in the two polynomials p_{n_1,n_2} and q_{n_1,n_2}. Hence a non trivial solution with $q_{n_1,n_2} \neq 0$ always exists. We put

$$e_{n_1,n_2}(z) = p_{n_1,n_2}(z)e^{-z/2} + q_{n_1,n_2}(z)e^{z/2},$$

so that

$$e_{n_1,n_2}(z) = \mathcal{O}\left(\omega_{n_1+n_2+1}(z)\right).$$

Note that when all the interpolation points $z_i^{(n_1+n_2)}$ are chosen to be equal to zero, we recover the usual Padé approximant of type (n_1, n_2) to e^z.

In this paper, we will stick to the diagonal case, i.e. $n_1 = n_2 = n$, and we will simply write p_n and q_n for $p_{n,n}$ and $q_{n,n}$, respectively. It is not difficult to check that the rational function p_n/q_n associated to a pair (p_n, q_n) solving an equation of the type (1.1), where e^z can be replaced by any function *f* defined in a subset of $\mathbb{C}$ where interpolation takes place, is always unique. On the contrary, the pairs (p_n, q_n) are in general not unique, even when normalized by some multiplicative constant.

We will be interested in the asymptotic behavior of the rational interpolants p_n and q_n. Throughout, we will assume that the (complex) interpolation points in $B^{(2n)}$ have a maximum modulus which grows at most like $c \log n$, where $c < 1/2$ is some given positive constant. It will be a consequence of our analysis that for *n* large, the solution of type (n, n) to (1.1) satisfies

$$\deg p_n = n, \quad \deg q_n = n.$$

In particular, for *n* large, a solution normalized so that q_n is monic will be unique.

The main task of the paper will be the study of the asymptotics of the scaled diagonal polynomials

$$P_n(z) = p_n(2nz), \qquad Q_n(z) = q_n(2nz) \tag{1.2}$$

and of the remainder term

$$E_n(z) = P_n(z)e^{-nz} + Q_n(z)e^{nz} = \mathcal{O}\left(\Omega_n(z)\right), \tag{1.3}$$

where

$$\Omega_n(z) = (2n)^{-2n-1}\omega_{2n+1}(2nz).$$

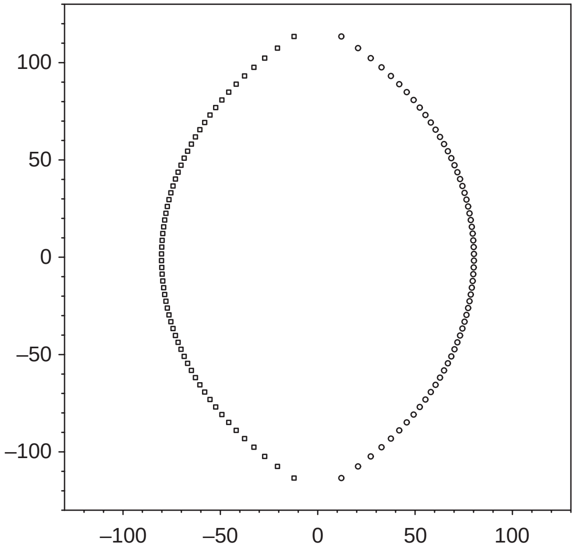

Fig. 1. Zeros of the diagonal Padé polynomials p_{60}^0 (the boxes on the left) and q_{60}^0 (the circles on the right).

We will always choose the normalization in (1.3) so that Q_n is a monic polynomial. From these asymptotics, the limit distributions of the zeros of P_n, Q_n, and E_n will follow. The zeros of P_n and Q_n accumulate on two specific arcs in the complex plane, symmetric with respect to the imaginary axis, while the zeros of E_n accumulate on two segments, on the imaginary axis itself. To illustrate, the zeros of the usual Padé approximants p_n^0, q_n^0 have been plotted for the value $n = 60$ in Fig. 1. The picture shows their particular distributions, first observed and studied by Saff and Varga in [37].

We will prove that the zeros of the scaled rational interpolants P_n and Q_n and the zeros of the scaled Padé approximants P_n^0 and Q_n^0 share the same asymptotic distribution. In other words, the geometry which underlies the limiting behavior of the zeros remains unchanged when considering general rational interpolants instead of plain Padé approximants.

The Padé approximants (and more generally simultaneous Padé approximants) to the exponential function were first studied by Hermite [19] in connection with his proof of the transcendance of e. Then, Padé, a student of Hermite, proved their convergence to e^z, uniformly in the complex plane, by making use of the explicit formulas, originally determined by Hermite,

$$p_{n_1,n_2}^0(z)=\sum_{j=0}^{n_1}\frac{(n_1+n_2-j)!n_1!z^j}{(n_1+n_2)!j!(n_1-j)!},\quad q_{n_1,n_2}^0(z)=\sum_{j=0}^{n_2}\frac{(n_1+n_2-j)!n_2!(-z)^j}{(n_1+n_2)!j!(n_2-j)!},$$

see [30–32] and [33, Section 75]. He obtained in this way a very nice, but rare, property of these approximants, since uniform convergence of the Padé approximants is known to happen only for a very few classes of functions in the complex plane.

Rational interpolation or equivalently, multipoint Padé approximation is an old subject whose study goes back at least to Cauchy and Jacobi, see [9,20]. Usually, the theory is divided into two parts, on one hand, an algebraic part concerned with recurrence relations, determinantal identities and algorithms for computations, and on another hand, an analytic part concerned with convergence aspects. The theorem of uniform convergence derived in this paper for the exponential function relates to the second part. It is of the same nature as the generalization that was obtained in [18] for multipoint Padé approximants to Markov functions. Note that contrary to the hypothesis made in [18], no symmetry assumption is made on the set of interpolation points. In connection with the problem of the limiting distribution of zeros, the present results may also be seen as closely related to the study initiated by Szegő in [42] concerning the distribution of the zeros of Taylor sections of the series for e^z, subsequently generalized by Saff and Varga in [36–38] to the zeros of the Padé approximants to e^z (see also [16,44]), and more recently by Stahl in [39,40] to the zeros of the quadratic Hermite–Padé approximants to e^z.

It may come as a surprise that Padé's result for the exponential function was not generalized to more general scheme of interpolation points in the complex plane until recently. It was only in [6] that this property was obtained for uniformly bounded sets of interpolation points on the real axis. There, the main ingredients were Rolle's theorem and results from the geometry of polynomials. Subsequently, the same property was also proved for some cases of conjugate interpolation points in a given compact set of $\mathbb{C}$, essentially by using an analog of Rolle's theorem for real exponential polynomials in the complex plane, see [45]. Nevertheless, to handle the case of complex interpolation points in full generality, it seemed necessary to introduce some new idea.

We follow an approach similar to the one used in [27] for studying quadratic Hermite–Padé approximants to e^z. The asymptotic analysis is based on a Riemann–Hilbert formulation for the polynomials P_n and Q_n, combined with a steepest descent analysis for Riemann–Hilbert problems. This technique originated with Deift and Zhou [15] and is currently applied to problems in such many different areas as integrable systems [14], random matrix theory [11,12], combinatorics [3] and orthogonal polynomials [7,12,13,21]. The book [10] and the lecture notes [22] are excellent introductions to the technique. Refs. [8,5,24,25] give further orientation about recent developments.

Note that [1,2] contain a thorough study of strong asymptotics of orthogonal polynomials with varying complex weights. Such orthogonal polynomials naturally arise in the context of rational approximation problems. In this connection, Theorem 2 of [1] is especially relevant here, which shows that strong asymptotics can be derived as soon as the geometry of the problem (more precisely the support of an equilibrium measure with respect to an extremal problem in potential theory with external field) is known and satisfies a symmetry and a connectedness assumptions. Note also that our asymptotic results concerning the scaled rational interpolants, when specialized to the Padé case, are not new since the polynomials p_n^0 and q_n^0 agree in this case, up to some multiplicative constant, with the Laguerre polynomials $L_n^{(-2n-1)}$ with negative parameter $-2n-1$, see [23] where the scaled polynomials $L_n^{(-2n-1)}(nz)$ were studied through the formulation of a Riemann–Hilbert problem.

Here, the Riemann–Hilbert problem is to find a 2×2 matrix valued function $Y:\mathbb{C}\setminus\Gamma\to\mathbb{C}^{2\times 2}$ where Γ is a closed contour in the complex plane encircling the origin (hence all the points $z_i^{(2n)}/2n$ for n large) once in the positive direction, such that

1. Y is analytic in $\mathbb{C} \setminus \Gamma$.
2. Y satisfies the jump condition

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & \Omega_n^{-1}(z)e^{-2nz} \\ 0 & 1 \end{pmatrix}, \quad z \in \Gamma, \tag{1.4}$$

   where $Y_+(z)$ and $Y_-(z)$ denote the limiting values of $Y(z')$ as z' approaches $z \in \Gamma$ from the inside and outside of Γ, respectively.
3. For large z

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^{n+1} & 0 \\ 0 & z^{-n-1} \end{pmatrix}, \quad z \to \infty. \tag{1.5}$$

We will show in Section 5.1 that the Riemann–Hilbert problem has a unique solution for n large, and that $Y_{22}(z) = \Omega_n^{-1}(z)Q_n(z)$ for z outside Γ, $Y_{22}(z) = \Omega_n^{-1}(z)e^{-nz}E_n(z)$ for z inside Γ, and $Y_{21}(z) = P_n(z)$.

The steepest descent analysis consists of a number of transformations. A crucial role is played by the Riemann surface defined by

$$z = \frac{w}{(w^2 - 1/4)}, \tag{1.6}$$

which is considered as a two sheeted surface with a cut along an arc Γ_P. The jumps of the two inverse mappings of (1.6) across the arc determine a probability measure μ_P supported on Γ_P. This measure turns out to be the limiting distribution of the normalized zero counting measures of the polynomials P_n.

We choose the closed contour Γ in the Riemann–Hilbert problem for Y so that it contains the cut Γ_P. The measure μ_P and its g-transform

$$g_P(z) = \int \log(z - s)\, d\mu_P(s)$$

are used to make the first transformation of the Riemann–Hilbert problem, which has the effect of normalizing the problem at infinity. Then we follow the general scheme, as presented in [13] or [10], for the asymptotic analysis of Riemann–Hilbert problems. It leads to a final Riemann–Hilbert problem whose solution has an explicit asymptotic behavior for $n \to \infty$, see Theorem 6.4 in Section 6.3. Tracing our steps back to the original Riemann–Hilbert problem, we obtain strong asymptotic formulas for the scaled multipoint Padé approximants in every region of the complex plane. In particular, we obtain asymptotic formulas on the sets where the zeros are and their endpoints. These last results, though not necessary in deriving uniform convergence of the rational interpolants, are of independent interest.

In Section 2 we state the normality property of the rational interpolation problem to e^z, uniform convergence of the polynomials p_n and q_n, and an estimate for the remainder term e_n, which follow readily from the asymptotic results for the scaled polynomials P_n, Q_n and for the remainder E_n. The asymptotic results make use of the functions obtained from the Riemann surface. The Riemann surface and the measures and functions derived from it are also described in Section 2. In Section 3 we prove the statements about the Riemann surface and other geometrical objects involved in the problem. To prepare for the transformations of the Riemann–Hilbert problem we need relations between the various functions involved, such as the inverse mappings of (1.6) and the function g_P. These properties are established in Section 4. Sections 5 and 6 contain the transformations of the Riemann–Hilbert problem and the asserted asymptotic results are proven in Section 7.

The general layout of the paper is similar to that of [27]. Actually, the method to derive the strong asymptotics of the scaled rational interpolants follows closely the one used in obtaining the strong asymptotics of the quadratic Hermite–Padé approximants to the exponential function. The analysis here is slightly simpler since we only need 2×2 matrices instead of the 3×3 matrices that were necessary in [27]. At the same time, the analysis demands at different places special care as we work with rational interpolants, instead of plain Padé approximants. Some parts of the proofs have not been displayed, especially those that can be adapted from the corresponding proofs in [27]. In this respect, it may be helpful as reading the present paper to have Ref. [27] at hand. Besides, Ref. [26] gives a very brief overview of [27].

2. Statement of results

Throughout, the following hypothesis on the scheme B of interpolation points will be made.

There exists $0 < \alpha \leqslant 1$ and $c > 0$ such that, for each n, the points of $B^{(2n)}$ lie in a closed disk $\mathcal{D}_n$ centered at the origin, of a radius ρ_n satisfying the growth condition

$$\forall n > 0, \quad \rho_n \leqslant \left(\frac{1-\alpha}{2}\right) \log n + c. \tag{2.1}$$

2.1. Normality of the rational interpolation problem

We first state a result which says that, given a sequence of closed disks $\mathcal{D}_n$ whose radius meets condition (2.1), an exponential polynomial $p_n(z) + q_n(z)e^z$ cannot have more than $2n + 1$ zeros in $\mathcal{D}_n$ for n large. In the terminology of Padé approximation, when all $\mathcal{D}_n$ reduces to the origin, such a property is usually rephrased by saying that the problem under study is normal. We will keep this terminology here.

Theorem 2.1. *Let $B := B^{(2n)} = \{z_i^{(2n)}\}_{i=0}^{2n}$, be a given triangular sequence of complex interpolation points satisfying (2.1). Then, there exists some integer $N > 0$ such that for $n \geqslant N$, the rational interpolants p_n and q_n of type (n, n) to e^z satisfying*

$$p_n(z) + q_n(z)e^z = \mathcal{O}\left(\omega_{2n+1}(z)\right), \quad \mathit{where} \quad \omega_{2n+1}(z) = \prod_{i=0}^{2n}\left(z - z_i^{(2n)}\right), \tag{2.2}$$

have full degrees, namely

$$\deg p_n = n, \quad \deg q_n = n.$$

In particular, a rational interpolant normalized so that q_n is monic is unique.

Theorem 2.1 will be deduced from the strong asymptotics for the scaled rational interpolants to be given in Theorem 2.10.

Remark. Normality does not hold for any degree. Indeed, it suffices to choose the closed disk centered at the origin, of radius 2π, and $\{-2i\pi, 0, 2i\pi\}$ as three interpolation points.

Then, $q_n(z) = 1$ and $p_n(z) = -1$ solve the corresponding interpolation problem with $\deg p_n < 1$ and $\deg q_n < 1$.

2.2. Local uniform convergence of the diagonal rational interpolants

Let us state now the result which is the main goal of our study, that is uniform convergence in the complex plane of the rational interpolants to the exponential function. Actually, we will prove more than that, namely, as in the Padé case, separated convergence of the numerator and denominator of the interpolants, see (2.3). Moreover, we will also obtain a sharp estimate for the error functions $e^z + r_n(z)$, see (2.4).

Theorem 2.2. *Let B be a scheme of points satisfying* (2.1) *and let p_n and q_n be the rational interpolants of type (n, n) to e^z such that* (2.2) *holds. Then, the following three assertions hold true*:

(i) *All the zeros and poles of $r_n = p_n/q_n$ tend to infinity, as n becomes large, sufficiently fast so that no poles of r_n lie in the disk $\mathcal{D}_n$, for n large. Hence, dividing the first equation in* (2.2) *by q_n, we get r_n as a true rational interpolant to e^z satisfying*

$$e^z + r_n(z) = \mathcal{O}\,(\omega_{2n+1}(z)).$$

(ii) *As $n \to \infty$,*

$$r_n(z) \to -e^z, \quad p_n(z) \to -e^{z/2}, \quad q_n(z) \to e^{-z/2}, \tag{2.3}$$

locally uniformly in $\mathbb{C}$, where q_n is normalized so that $q_n(0) = 1$.

(iii) *for n large,*

$$e^z + r_n(z) = (-1)^{n+1}\left(\frac{e}{4n}\right)^{2n+1}\omega_{2n+1}(z)e^{z-1}\left(1 + \mathcal{O}\left(\frac{1}{n^\alpha}\right)\right), \tag{2.4}$$

locally uniformly in $\mathbb{C}$.

As Theorem 2.1, Theorem 2.2 follows easily from the strong asymptotics given in Theorem 2.10.

Remark. Theorem 2.2 generalizes for the diagonal case results about rational interpolation of the exponential function that were obtained with real interpolation points and some cases of complex interpolation points in Theorems 2.1 and 2.2 of [6] and Theorems 2.1 and 2.3 of [45], respectively.

2.3. The Riemann surface

In order to state our convergence results for the scaled rational interpolants P_n and Q_n, we first introduce an appropriate Riemann surface. The starting point for our analysis will be the explicit integral formulas for the scaled Padé approximants P_n^0 and Q_n^0, which are

$$P_n^0(z) = \frac{Ce^{nz}}{2\pi i}\oint_{C_{-1/2}} \frac{e^{2nzw}\,dw}{(w^2 - 1/4)^{n+1}}, \tag{2.5}$$

$$Q_n^0(z) = \frac{Ce^{-nz}}{2\pi i} \oint_{C_{1/2}} \frac{e^{2nzw}\,dw}{(w^2-1/4)^{n+1}}. \tag{2.6}$$

Here C_j is a closed contour in the complex plane encircling j in the positive direction, which does not encircle the other point in $\{-1/2, 1/2\}$. The number C in (2.5) and (2.6) is a normalization constant. The Riemann surface is chosen so that it parameterizes the critical points of the function

$$w \mapsto 2zw - \log\left(w^2 - 1/4\right). \tag{2.7}$$

Note that the integrals in formulas (2.5) and (2.6) have the form

$$\oint_{C_j} \frac{1}{(w^2-1/4)} e^{n\left(2zw - \log\left(w^2-1/4\right)\right)}\,dw \tag{2.8}$$

and that by the classical saddle point analysis for the asymptotic evaluation of integrals, the main contribution to the integral (2.8) comes from a critical point of (2.7). So we define $\mathcal{R}$ as the Riemann surface for the function

$$z = z(w) = \frac{1}{2}\left(\frac{1}{w-1/2} + \frac{1}{w+1/2}\right) = \frac{w}{(w^2-1/4)}. \tag{2.9}$$

Note that we obtain (2.9) if we set the derivative of (2.7) equal to zero and solve for z. The rational function (2.9) has two inverse mappings. These are the two solutions of the quadratic equation

$$zw^2 - w - \frac{z}{4} = 0. \tag{2.10}$$

The Riemann surface $\mathcal{R}$ consists of two sheets $\mathcal{R}_P$, and $\mathcal{R}_Q$, see Proposition 2.3. The bijective mapping $\psi : \mathcal{R} \to \overline{\mathbb{C}}$ is the inverse of (2.9). We denote its restriction to the two sheets by ψ_P, and ψ_Q, respectively. So $\psi_P(z)$, and $\psi_Q(z)$ are the two solutions of (2.10), given explicitly by

$$\psi_P(z) = \frac{1-\sqrt{1+z^2}}{2z}, \quad \psi_Q(z) = \frac{1+\sqrt{1+z^2}}{2z}, \tag{2.11}$$

where the square root is chosen to be positive for large positive z. Typically we will identify the two sheets with copies of the complex plane, and so ψ_P and ψ_Q are defined on $\mathbb{C}$ with an appropriate cut Γ_P connecting the two branch points

$$z_1 = i, \quad z_2 = -i,$$

of the Riemann surface. The sheets $\mathcal{R}_P$ and $\mathcal{R}_Q$ are glued together along the cut Γ_P.

The two branch points $z_1 = z(w_1)$, $z_2 = z(w_2)$ are related to the points w_1, w_2 for which $z'(w) = 0$, namely

$$w_1 = -i/2, \quad w_2 = i/2. \tag{2.12}$$

The precise sheet structure of $\mathcal{R}$ is given in the following proposition.

Proposition 2.3. *There is an analytic curve Γ_P from z_1 to z_2 lying in the left half-plane, such that the following hold.*

(a) *Two inverse mappings ψ_P and ψ_Q of (2.9) exist so that ψ_P and ψ_Q are defined and analytic on $\mathbb{C} \setminus \Gamma_P$.*

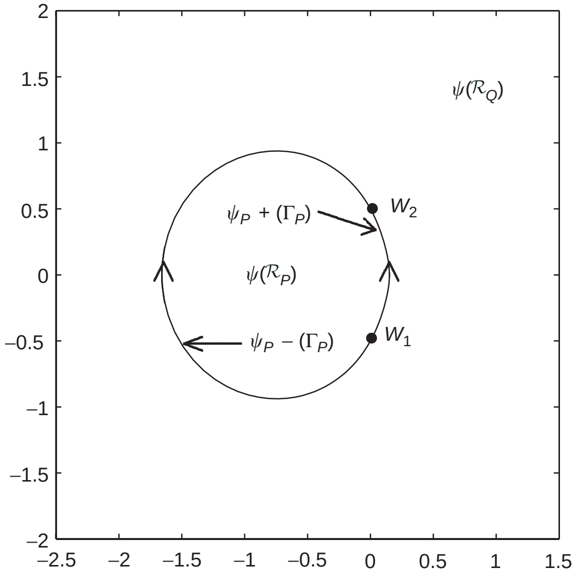

Fig. 2. ψ-image of the Riemann surface $\mathcal{R}$.

(b) *At infinity, we have the values* $\psi_P(\infty) = -1/2$, *and* $\psi_Q(\infty) = 1/2$.
(c) *For* $z \in \Gamma_P$, *we have*

$$\frac{1}{2\pi i}\int_{z_1}^{z} (\psi_Q - \psi_P)_+(s)\, ds \in \mathbb{R}, \tag{2.13}$$

with integration along the $+$*side of* Γ_P. [*This is the side of* Γ_P *that is on the left while going from* z_1 *to* z_2 *along* Γ_P.]

The functions ψ_P, ψ_Q are defined on the P, and Q sheet of $\mathcal{R}$, respectively. Together they constitute a conformal map from $\mathcal{R}$ onto the Riemann sphere. The images of the two sheets are shown in Fig. 2.

Other curves of interest to our problem are defined by the property that $\frac{1}{2\pi i}\int_{z_1}^{z}(\psi_Q - \psi_P)(s)\, ds$ is real. These curves are described by the following proposition, see Fig. 3.

Proposition 2.4. *There are four analytic curves where* $\frac{1}{2\pi i}\int_{z_1}^{z}(\psi_Q - \psi_P)(s)\, ds$ *is real. One of them is* Γ_P, *a second one is the mirror image of* Γ_P *with respect to the imaginary axis. We call this curve* Γ_Q. *The other two curves are semi–infinite segments lying on the imaginary axis. They join* z_1 *and* z_2 *with infinity, and we call them* $\Gamma_{E,1}$ *and* $\Gamma_{E,2}$, *respectively.*

All contours are oriented as shown in Fig. 3. The orientation induces a $+$side and a $-$side on each contour, where the $+$side is on the left and the $-$side on the right while traversing the contour according to its orientation. Propositions 2.3 and 2.4 are proved in Section 3.

We also define the contour

$$\Gamma_E = \Gamma_{E,1} \cup \Gamma_{E,2}. \tag{2.14}$$

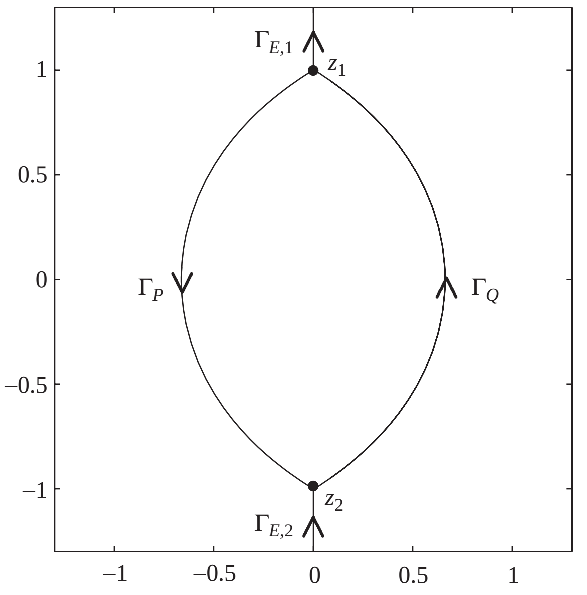

Fig. 3. Curves for which $\frac{1}{2\pi i}\int_{z_1}^{z}(\psi_Q-\psi_P)(s)\,ds$ is real.

The contours Γ_P, Γ_Q, and Γ_E divide the complex plane into three domains. We denote the unbounded domains by $D_{\infty,P}$, $D_{\infty,Q}$, as shown in Fig. 4. The bounded domain is denoted by D_0, see also Fig. 4. Moreover, we put

$$D_\infty=\left(D_{\infty,P}\cup D_{\infty,Q}\cup(\Gamma_E\setminus\{z_1,z_2\})\right). \tag{2.15}$$

This is the unbounded domain bounded by Γ_P and Γ_Q.

2.4. *The measures μ_P, μ_Q, and μ_E*

We now define measures on the curves Γ_P, Γ_Q, and Γ_E. The complex line element *ds* is defined according to the orientation of these curves given in Fig. 4.

Definition 2.5. We define a measure μ_P on Γ_P and a measure μ_Q on Γ_Q by

$$\begin{aligned} d\mu_P(s) &= \frac{1}{\pi i}\,(\psi_Q-\psi_P)_+(s)\,ds, \quad s\in\Gamma_P,\\ d\mu_Q(s) &= \frac{1}{\pi i}\,(\psi_Q-\psi_P)_+(-s)\,ds, \quad s\in\Gamma_Q \end{aligned} \tag{2.16}$$

and a measure μ_E on Γ_E by

$$d\mu_E(s)=\frac{1}{\pi i}\,(\psi_Q-\psi_P)(s)\,ds, \quad s\in\Gamma_{E,1}\cup\Gamma_{E,2}. \tag{2.17}$$

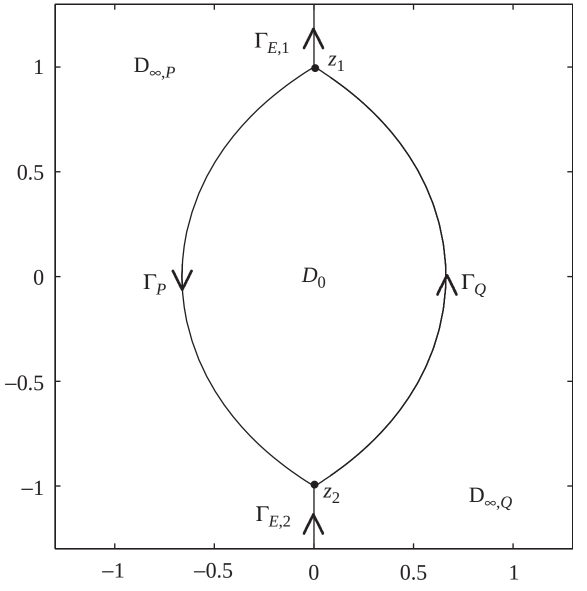

Fig. 4. Curves Γ_P, Γ_Q, Γ_E, and domains D_0, $D_{\infty,P}$, $D_{\infty,Q}$.

Theorem 2.6. *We have that μ_P is a probability measure on Γ_P and μ_Q is a probability measure on Γ_Q. The measure μ_E is a positive measure on Γ_E.*

Theorem 2.6 is proved in Section 4.1.

The relevance of these measures is shown by the following theorem. For every polynomial p of exact degree n, we denote by v_p the normalized zero counting measure. Thus

$$v_p = \frac{1}{n} \sum_{p(z)=0} \delta_z$$

where each zero is counted according to its multiplicity. We also define a zero counting measure for the remainder function E_n, namely

$$v_{E_n} = \frac{1}{n} \sum_{\substack{E_n(z)=0 \\ \Omega_n(z)\neq 0}} \delta_z,$$

where the normalization by n now corresponds to the degree of approximation and the $2n+1$ interpolatory zeros of E_n at the roots of Ω_n have been excluded.

Theorem 2.7. *We have*

$$v_{P_n} \stackrel{*}{\to} \mu_P, \qquad v_{Q_n} \stackrel{*}{\to} \mu_Q, \tag{2.18}$$

where the convergence is in the sense of weak* *convergence of measures, i.e.,* $\mu_n \xrightarrow{*} \mu$ *if* $\int f\, d\mu_n \to \int f\, d\mu$ *for every bounded continuous function f. Furthermore, we have*

$$\nu_{E_n} \xrightarrow{*} \mu_E, \tag{2.19}$$

in the sense that

$$\lim_{n\to\infty} \int f(s)\, d\nu_{E_n}(s) = \int f(s)\, d\mu_E(s)$$

for every continuous function f such that $f(s) = \mathcal{O}(s^{-2})$ *as* $s \to \infty$.

In contrast to the measures ν_{P_n} and ν_{Q_n} which are probability measures, the measures ν_{E_n} have infinite mass. They also have unbounded support. As a result, the proof of the limit (2.19) is more involved than that of (2.18). A sketch of the proof of Theorem 2.7 is given in Section 7.3. For details, the reader is referred to the proof of similar results derived in Theorem 2.5 of [27].

2.5. *The g-function g_P and the function φ_P*

For the strong asymptotic results we need the log-transform (or complex logarithmic potential) of the measure μ_P.

Definition 2.8. We introduce the function

$$g_P(z) = \int_{\Gamma_P} \log(z-s)\, d\mu_P(s), \qquad z \in \mathbb{C} \setminus \Gamma_P, \tag{2.20}$$

which is defined modulo $2\pi i$.

Thus g_P is a multivalued function, depending on the specific choice of the branches of the logarithmic functions. Our results will involve expressions like e^{ng_P}, and then the multivaluedness will play no role.

Another important function is the function φ_P given by

$$\varphi_P(z) = \int_{z_1}^{z} (\psi_Q - \psi_P)(s)\, ds. \tag{2.21}$$

The path of integration in (2.21) is in $\mathbb{C} \setminus (\Gamma_P \cup \{0\})$. The function φ_P is multivalued but the real part is well-defined. From Proposition 2.4 we know that $\operatorname{Re} \varphi_P = 0$ on the curves Γ_P, Γ_Q, $\Gamma_{E,1}$, and $\Gamma_{E,2}$. An important property of φ_P is given in the following lemma.

Lemma 2.9. *The real part of* φ_P *is zero exactly on* Γ_P, Γ_Q, $\Gamma_{E,1}$, *and* $\Gamma_{E,2}$. *The real part of* φ_P *is negative in* $D_{\infty,P} \cup D_0$, *and it is positive in the remaining part* $D_{\infty,Q}$ *of the plane.*

Lemma 2.9 is proved in Section 4.2.

2.6. *Strong asymptotics of the scaled rational interpolants*

Recall that inequality (2.1) has been assumed throughout, and that the two polynomials P_n and Q_n satisfy (1.3) where Q_n is monic. In the rest of the paper, the following function will be used

$$s_n(z) = z^{-2n-1}\Omega_n(z) = \prod_{i=0}^{2n}\left(1 - \frac{z_i^{(2n)}}{2nz}\right). \tag{2.22}$$

We have $s_n(\infty) = 1$, $n \geqslant 0$, and from (2.1), we see that, for any compact set K of $\overline{\mathbb{C}} \setminus \{0\}$ and for n large enough, s_n has no zeros in K.

We will also need the function $\sqrt{4w^2+1}$ which branches at the two points w_k given in (2.12). We choose as cut for this function the curve $\psi_{P+}(\Gamma_P)$ (see Fig. 2), and assume that it is positive for large positive w.

2.6.1. *Strong asymptotics away from the zeros*

The following theorem gives the strong asymptotics of the polynomials P_n, Q_n and the remainder term E_n away from their zeros.

Theorem 2.10. *With the functions defined above, we have*

$$P_n(z) = (-1)^n \frac{\sqrt{2}s_n(1/\psi_P(z))e^{ng_P(z)}}{s_n(2)\sqrt{4\psi_P^2(z)+1}}\left(1 + \mathcal{O}\left(\frac{1}{n^\alpha}\right)\right) \tag{2.23}$$

uniformly for z in compact subsets of $\mathbb{C} \setminus \Gamma_P$,

$$Q_n(z) = \begin{cases} (-1)^{n+1} \frac{\sqrt{2}s_n(1/\psi_P(z))e^{n(g_P(z)-2z)}}{s_n(2)\sqrt{4\psi_P^2(z)+1}}\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right) & \text{for } z \in D_0 \cup \Gamma_P \setminus \{z_1, z_2\} \\ \frac{\sqrt{2}z^{2n}s_n(1/\psi_Q(z))e^{-ng_P(z)}}{s_n(2)\sqrt{4\psi_Q^2(z)+1}}\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right) & \text{for } z \in \mathbb{C} \setminus \overline{D_0}, \end{cases} \tag{2.24}$$

uniformly for z in compact subsets of $\mathbb{C} \setminus \Gamma_Q$. *Furthermore we have*

$$E_n(z) = \begin{cases} (-1)^{n} \frac{\sqrt{2}s_n(1/\psi_P(z))e^{n(g_P(z)-z)}}{s_n(2)\sqrt{4\psi_P^2(z)+1}}\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right) & \text{for } z \in D_{\infty,P} \cup \Gamma_P \setminus \{z_1, z_2\} \\ \frac{\sqrt{2}\Omega_n(z)e^{n(z-g_P(z))}}{zs_n(2)s_n(1/\psi_P(z))\sqrt{4\psi_Q^2(z)+1}}\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right) & \text{for } z \in \mathbb{C} \setminus \overline{D_{\infty,P}}, \end{cases} \tag{2.25}$$

uniformly for z in compact subsets of $\mathbb{C} \setminus \Gamma_E$.

Remark. The asymptotic formula (2.24) for the polynomial Q_n may as well be stated using functions which are the analytic continuations of ψ_P and ψ_Q to $\mathbb{C} \setminus \Gamma_Q$. Let $\widetilde{\psi}_P$ and $\widetilde{\psi}_Q$ be the functions in (2.11) where the cut for the square root is now chosen to be Γ_Q. Hence, $\widetilde{\psi}_P(z) = \psi_P(z)$ and $\widetilde{\psi}_Q(z) = \psi_Q(z)$ for $z \in \mathbb{C} \setminus \overline{D_0}$, while $\widetilde{\psi}_P(z) = \psi_Q(z)$ and $\widetilde{\psi}_Q(z) = \psi_P(z)$ for $z \in D_0$. Moreover, let

$$\begin{aligned} g_Q(z) &= \int_{\Gamma_Q} \log(z-s)\, d\mu_Q(s), \qquad z \in \mathbb{C} \setminus \Gamma_Q, \\ &= \int_{\Gamma_Q} \log(z-s)(\widetilde{\psi}_Q - \widetilde{\psi}_P)_+(s)\, ds, \end{aligned}$$

where the measure $d\mu_Q$ is defined in (2.16). Then, one checks that $g_Q(z) = g_P(-z) - \log(-1)$. Using the functions $\widetilde{\psi}_P$, $\widetilde{\psi}_Q$, g_Q and the square root $\sqrt{4w^2+1}$ with a cut on the curve $\widetilde{\psi}_{Q+}(\Gamma_Q) = -\psi_{P+}(\Gamma_P)$, the two formulas in (2.24) rewrite more simply as the single one

$$Q_n(z) = \frac{\sqrt{2}s_n(1/\widetilde{\psi}_Q(z))e^{ng_Q(z)}}{s_n(2)\sqrt{4\widetilde{\psi}_Q^2(z)+1}}\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right), \tag{2.26}$$

uniformly for z in compact subsets of $\mathbb{C}\setminus\Gamma_Q$. Similarly, formula (2.25) for the error function E_n may also be translated in terms of $\widetilde{\psi}_P$, $\widetilde{\psi}_Q$ and g_Q.

Corollary 2.11. *For n large, the leading coefficient α_n of P_n, that is, by Theorem 2.1, the coefficient of degree n of P_n, satisfies the following asymptotics*:

$$\alpha_n = s_n^{-2}(2)\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right), \tag{2.27}$$

as $n \to \infty$.

2.6.2. Asymptotics near the curves Γ_P, Γ_Q, and Γ_E

It is also possible to obtain uniform asymptotics near the curves Γ_P, Γ_Q, and Γ_E, as well as in neighborhoods of the branch points z_1 and z_2. We have asymptotic formulas on Γ_P, Γ_Q and Γ_E, respectively, away from the branch points, which now consist of two terms.

Theorem 2.12. *Uniformly for z in compact subsets of the domain $(\Gamma_P\setminus\{z_1,z_2\})\cup D_{\infty,P}\cup D_0$, we have*

$$P_n(z) = (-1)^n\,\frac{\sqrt{2}s_n(1/\psi_P(z))e^{ng_P(z)}}{s_n(2)}\left[\frac{1}{\sqrt{4\psi_P^2(z)+1}}\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right)\right.$$
$$\left.\pm\frac{e^{2n\varphi_P(z)}}{\sqrt{4\psi_Q^2(z)+1}}\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right)\right]. \tag{2.28}$$

where the $+$sign holds in D_0 and the $-$sign holds in $D_{\infty,P}$.

Uniformly for z in compact subsets of the domain $\left(\Gamma_Q\setminus\{z_1,z_2\}\right)\cup D_{\infty,Q}\cup D_0$, we have

$$Q_n(z) = -(-1)^n\,\frac{\sqrt{2}e^{n(g_P(z)-2z)}}{s_n(2)}\left[\frac{s_n(1/\psi_P(z))}{\sqrt{4\psi_P^2(z)+1}}\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right)\right.$$
$$\left.-\frac{e^{2n\varphi_P(z)}s_n(1/\psi_Q(z))}{\sqrt{4\psi_Q^2(z)+1}}\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right)\right]. \tag{2.29}$$

Uniformly for z in compact subsets of the domain $(\Gamma_E \setminus \{z_1, z_2\}) \cup D_{\infty,P} \cup D_{\infty,Q}$, *we have*

$$E_n(z) = (-1)^n \frac{\sqrt{2}e^{n(g_P(z)-z)}}{s_n(2)} \left[\frac{s_n(1/\psi_P(z))}{\sqrt{4\psi_P^2(z)+1}} \left(1 + \mathcal{O}\left(\frac{1}{n^\alpha}\right)\right) \right.$$
$$\left. + \frac{e^{2n\varphi_P(z)} s_n(1/\psi_Q(z))}{\sqrt{4\psi_Q^2(z)+1}} \left(1 + \mathcal{O}\left(\frac{1}{n^\alpha}\right)\right) \right]. \tag{2.30}$$

Remark. For z away from Γ_P so that Re $\varphi_P(z) < 0$, the asymptotic formula (2.28) reduces to (2.23). On Γ_P we have Re $\varphi_P(z) = 0$, and then the two terms in (2.28) are of comparable magnitude. For $z \in \Gamma_P$, (2.28) can be re-written as

$$P_n(z) = (-1)^n \frac{\sqrt{2}s_n(1/\psi_{P-}(z))}{s_n(2)} \left[\frac{e^{ng_{P-}(z)}}{\sqrt{4\psi_{P-}^2(z)+1}} \left(1 + \mathcal{O}\left(\frac{1}{n^\alpha}\right)\right) \right.$$
$$\left. \pm \frac{e^{ng_{P+}(z)}}{\sqrt{4\psi_{P+}^2(z)+1}} \left(1 + \mathcal{O}\left(\frac{1}{n^\alpha}\right)\right) \right]. \tag{2.31}$$

Note that the asymptotic formula (2.29) (resp. (2.30)) holds in particular on Γ_Q (resp. Γ_E), away from the branch points z_1 and z_2, that is, on the curve where the zeros of Q_n (resp. E_n) accumulate.

It may be checked that the two terms in (2.29) are analytic continuations of the two different asymptotic formulas we have in (2.24). For $z \in D_{\infty,Q}$, we have $g_P(z) - 2z + 2\varphi_P(z) = -g_P(z) + 2\log z + 2\ell$ by (4.6), and then (2.29) reduces to the second formula in (2.24) since Re $\varphi_P(z) > 0$ in $D_{\infty,Q}$. For $z \in D_0$, we have Re $\varphi_P(z) < 0$ and we obtain the first formula in (2.24).

On Γ_Q the two terms in (2.29) have comparable absolute values. This causes the zeros of Q_n to be close to Γ_Q. Similarly, on Γ_E the two terms in (2.30) have comparable absolute values, which causes the zeros of E_n in D_∞ to be close to Γ_E.

2.6.3. Asymptotics near the branch points

Near the branch points, the asymptotic formulas involve the Airy function Ai, which is the unique solution of the differential equation

$$y''(z) = zy(z),$$

satisfying

$$\mathrm{Ai}(z) = \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3}z^{3/2}} \left(1 + \mathcal{O}\left(\frac{1}{z^{3/2}}\right)\right), \tag{2.32}$$

$$\mathrm{Ai}'(z) = \frac{-1}{2\sqrt{\pi}} z^{1/4} e^{-\frac{2}{3}z^{3/2}} \left(1 + \mathcal{O}\left(\frac{1}{z^{3/2}}\right)\right), \tag{2.33}$$

as $z \to \infty$ with $|\arg z| < \pi$, where $z^{1/4}$ and $z^{3/2}$ are defined with principal branch, see [29] for more details. We only deal with the asymptotic behavior of P_n, Q_n and E_n near z_1. Similar results can be given for the behavior near the other branch point z_2. Here we will take the branch of the function $\varphi_P(z)$ which is 0 at $z = z_1$. So it behaves like

$$\varphi_P(z) = c(z - z_1)^{3/2} + \mathcal{O}\left((z - z_1)^{5/2}\right)$$

as $z \to z_1$, with $c \neq 0$. Then we define the function

$$f_1(z) = \left[\frac{3}{2}\,\varphi_P(z)\right]^{2/3} \tag{2.34}$$

which is analytic for z in a neighborhood of z_1. We take that 2/3rd power so that $f_1(z)$ is real and negative for $z \in \Gamma_P$. Then

$$f_1(z) = c_1(z - z_1) + \mathcal{O}\left((z - z_1)^2\right) \tag{2.35}$$

as $z \to z_1$ with some constant $c_1 \neq 0$. Explicit calculations show that

$$c_1 = f_1'(z_1) = 2^{1/3} e^{-\frac{\pi i}{6}}. \tag{2.36}$$

In order to state the asymptotics near the branch points, we furthermore need to define two functions $\widetilde{s}_n$ and $\widehat{s}_n$ which are slight perturbations of the function s_n. To each interpolation point $z_i^{(2n)}$ of the scheme $B^{(2n)}$, we associate the point

$$\widetilde{z}_i^{(2n)} = \frac{2z_i^{(2n)}}{1 + \sqrt{1 + \frac{z_i^{(2n)2}}{4n^2}}}$$

and we define $\widetilde{s}_n$ and $\widehat{s}_n$ as

$$\widetilde{s}_n(z) = \prod_{i=0}^{2n}\left(1 - \frac{\widetilde{z}_i^{(2n)}}{2nz}\right), \quad \widehat{s}_n(z) = \prod_{i=0}^{2n}\left(1 - \frac{z_i^{(2n)}}{2nz} + \frac{z_i^{(2n)}\widetilde{z}_i^{(2n)}}{16n^2}\right). \tag{2.37}$$

Note that (2.1) implies that locally uniformly, as $n \to \infty$,

$$\widetilde{s}_n(z) = s_n(z)\left(1 + \mathcal{O}\left(\frac{\log^3 n}{n^2}\right)\right), \quad \widehat{s}_n(z) = s_n(z)\left(1 + \mathcal{O}\left(\frac{\log^2 n}{n}\right)\right). \tag{2.38}$$

If the parameter α in (2.1) equals 1, then the $\log n$ factor in the two previous $\mathcal{O}$ -term may be replaced with 1.

Theorem 2.13. *Let $\delta_n > 0$ be such that $\delta_n = \mathcal{O}(1/\rho_n^2)$ as $n \to \infty$. Then, uniformly for $|z - z_1| < \delta_n$,*

$$\begin{aligned}(-1)^{n+1} P_n(z) = {} & \sqrt{\pi} e^{(n+1)(g_P(z)+\varphi_P(z))} \\ & \times\Big[n^{1/6} h_1(z) \mathrm{Ai}\left((n+1)^{2/3} f_1(z)\right)\left(1 + \mathcal{O}\left(\frac{1}{n^\alpha}\right)\right) \\ & + n^{-1/6} h_2(z) \mathrm{Ai}'\left((n+1)^{2/3} f_1(z)\right)\left(1 + \mathcal{O}\left(\frac{1}{n^\alpha}\right)\right)\Big],\end{aligned} \tag{2.39}$$

$$
\begin{aligned}
&(-1)^{n+1}Q_n(z)\\
&\quad = -\sqrt{\pi}e^{2z}e^{(n+1)(g_P(z)+\varphi_P(z)-2z)}\\
&\qquad \times\Big[n^{1/6}h_1(z)e^{-2\pi i/3}\mathrm{Ai}\left(e^{-2\pi i/3}(n+1)^{2/3}f_1(z)\right)\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right)\\
&\qquad + n^{-1/6}h_2(z)e^{2\pi i/3}\mathrm{Ai}'\left(e^{-2\pi i/3}(n+1)^{2/3}f_1(z)\right)\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right)\Big]
\end{aligned}
\tag{2.40}
$$

and

$$
\begin{aligned}
&(-1)^{n+1}E_n(z)\\
&\quad = -\sqrt{\pi}e^{2z}e^{(n+1)(g_P(z)+\varphi_P(z)-2z)}\\
&\qquad \times\Big[n^{1/6}h_1(z)e^{2\pi i/3}\mathrm{Ai}\left(e^{2\pi i/3}(n+1)^{2/3}f_1(z)\right)\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right)\\
&\qquad + n^{-1/6}h_2(z)e^{-2\pi i/3}\mathrm{Ai}'\left(e^{2\pi i/3}(n+1)^{2/3}f_1(z)\right)\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right)\Big],
\end{aligned}
\tag{2.41}
$$

where h_1 and h_2 are two analytic functions in $|z-z_1|<\delta_n$, which have explicit expressions

$$h_1(z) = \left(N_{21}(z) + iz^{-1}e^{-2z}s_n(z)N_{22}(z)\right) f_1(z)^{1/4},$$

with the branch of the fourth root in $f_1(z)^{1/4}$ taken with a cut along Γ_P, and

$$h_2(z) = \left(-N_{21}(z) + iz^{-1}e^{-2z}s_n(z)N_{22}(z)\right) f_1(z)^{-1/4}.$$

Here

$$
\begin{aligned}
N_{21}(z) &= -\frac{\sqrt{2}\widetilde{s}_n(-2)\widehat{s}_n(1/\psi_P(z))e^{-g_P(z)}}{\sqrt{4\psi_P^2(z)+1}},\\
N_{22}(z) &= \frac{\sqrt{2}\widetilde{s}_n(-2)e^{g_P(z)}}{z\widetilde{s}_n(-4\psi_Q(z))\sqrt{4\psi_Q^2(z)+1}}.
\end{aligned}
$$

We use N_{21} and N_{22} to denote these functions, since in what follows they will appear as entries in a matrix N. Note that the functions h_1, h_2, N_{21} and N_{22} depend on n. Note also that the function $g_P+\varphi_P$ is analytic near $z=z_1$.

From the asymptotics near the branch points, one can deduce the behavior of the extreme zeros of P_n, Q_n and E_n near the branch points. We only state the result for the zeros of P_n, Q_n and E_n near z_1. Recall that the Airy function Ai has only negative real zeros, which we denote by $0>-\iota_1>-\iota_2>\cdots>-\iota_\nu>\cdots$.

Corollary 2.14. *Let*

$$
\beta(n) = \begin{cases} n^{-\alpha-2/3} & \textit{if } 0<\alpha<\frac{1}{3},\\ \frac{\log n}{n} & \textit{if } \frac{1}{3}\leqslant\alpha\leqslant 1,\\ \frac{1}{n} & \textit{if } \alpha=1. \end{cases}
\tag{2.42}
$$

Let $z^P_{v,n}$, $v = 1, \ldots, n$, be the zeros of P_n, ordered by increasing distance to z_1. Then for every $v \in \mathbb{N}$, we have

$$z^P_{v,n} = z_1 - \frac{\iota_v}{f_1'(z_1)}, n^{-2/3} + \mathcal{O}\left(\beta(n)\right) = z_1 + 2^{-1/3} e^{\frac{7}{6}\pi i}\iota_v n^{-2/3} + \mathcal{O}\left(\beta(n)\right), \tag{2.43}$$

as $n \to \infty$. Let $z^Q_{v,n}$, $v = 1, \ldots, n$, be the zeros of Q_n, ordered by increasing distance to z_1. Then for every $v \in \mathbb{N}$,

$$z^Q_{v,n} = z_1 - e^{2\pi i/3}\frac{\iota_v}{f_1'(z_1)} n^{-2/3} + \mathcal{O}\left(\beta(n)\right) = z_1 + 2^{-1/3} e^{-\frac{\pi i}{6}}\iota_v n^{-2/3} + \mathcal{O}\left(\beta(n)\right), \tag{2.44}$$

as $n \to \infty$. Let $z^E_{v,n}$, $v \geqslant 1$, be the zeros of E_n, ordered by increasing distance to z_1. Then for every $v \in \mathbb{N}$,

$$z^E_{v,n} = z_1 - e^{-2\pi i/3}\frac{\iota_v}{f_1'(z_1)} n^{-2/3} + \mathcal{O}\left(\beta(n)\right) = z_1 + 2^{-1/3} e^{\frac{\pi i}{2}}\iota_v n^{-2/3} + \mathcal{O}\left(\beta(n)\right), \tag{2.45}$$

as $n \to \infty$.

Remark. Note that the first two terms in the previous expansions do not depend on the actual choice of the interpolation points.

3. Geometry of the problem

3.1. Trajectories of quadratic differentials

Proof of Proposition 2.4. We recall that ψ_P and ψ_Q will be the two inverse mappings of

$$z = z(w) = \frac{w}{w^2 - 1/4},$$

given explicitly by (2.11), where the square root is defined outside a cut Γ_P connecting the two branch points $z_1 = i$ and $z_2 = -i$. We note that ψ_Q has a pole at 0 and from (2.11) it follows that

$$\psi_Q(z) = \frac{1}{z} + \mathcal{O}(z) \qquad \text{as } z \to 0. \tag{3.1}$$

Near infinity, we have from analyzing (2.9)

$$\psi_P(z) = -\frac{1}{2} + \frac{1}{2z} + \mathcal{O}\left(\frac{1}{z^2}\right), \tag{3.2}$$

$$\psi_Q(z) = \frac{1}{2} + \frac{1}{2z} + \mathcal{O}\left(\frac{1}{z^2}\right), \tag{3.3}$$

as $z \to \infty$. We study the curves so that $\frac{1}{2\pi i}\int_{z_1}^{z}(\psi_Q - \psi_P)(s)\,ds$ is real. If $z = z(t)$ is arclength parametrization of such a curve, then

$$\mathrm{Re}\left[\int_{z_1}^{z(t)}(\psi_Q - \psi_P)(s)\,ds\right] = 0,$$

which upon differentiating leads to $\mathrm{Re}\left[z'(t)(\psi_Q - \psi_P)(z(t))\right] = 0$. Since $z'(t) \neq 0$, and $(\psi_Q - \psi_P)(z) \neq 0$ (except at the branch points z_1 and z_2), we find that $-z'(t)^2(\psi_Q - \psi_P)(z(t))^2 > 0$. Thus the curve is what is known in geometric function theory as a trajectory of the quadratic differential

$$-(\psi_Q - \psi_P)^2(z)\,dz^2, \tag{3.4}$$

see [34,41]. Now, $(\psi_Q - \psi_P)^2(z)$ is well-defined in the left half-plane, irrespective of the exact choice for Γ_P. It is analytic with simple zeros at the branch points z_1 and z_2, and a double pole at 0.

Trajectories of the quadratic differential (3.4) which start from or end at z_1 or z_2 are called critical trajectories. From the local structure of trajectories of quadratic differentials, it is known that three trajectories emanate from a simple zero, with tangent directions at this point that divide the plane locally into three sectors of equal aperture $2\pi/3$. Since ψ_P and ψ_Q are real functions, that is $\psi_P(\bar{z}) = \overline{\psi_P(z)}$ and $\psi_Q(\bar{z}) = \overline{\psi_Q(z)}$, we deduce that the trajectories are symmetric with respect to the real axis. On the other hand, according to the location of the point z with respect to the cut Γ_P, we may have

$$\sqrt{1+(-z)^2} = \sqrt{1+z^2} \text{ whence } \psi_P(-z) = -\psi_P(z) \text{ and } \psi_Q(-z) = -\psi_Q(z)$$

or

$$\sqrt{1+(-z)^2} = -\sqrt{1+z^2} \text{ whence } \psi_P(-z) = -\psi_Q(z) \text{ and } \psi_Q(-z) = -\psi_P(z).$$

From this, we deduce that trajectories are symmetric with respect to the origin, hence also symmetric with respect to the imaginary axis. Consequently the directions of the three trajectories at z_1 can only have arguments $\pi/2$, $7\pi/6$, $11\pi/6$ or $\pi/6$, $5\pi/6$, $3\pi/2$. In the second case, the trajectory with initial direction $3\pi/2$ must reach the double pole at 0. Since locally near the origin, the quadratic differential is $i^2z^{-2}dz^2$, it is known that trajectories near this point are closed contours around it, see [34, p. 215]. Hence the second case cannot occur, so that the three trajectories emanating from z_1 have initial directions $\pi/2$, $7\pi/6$, $11\pi/6$. We label these three trajectories by $\Gamma_{E,1}$, Γ_P, and Γ_Q respectively. The trajectory $\Gamma_{E,1}$ can only connect z_1 with infinity along the imaginary axis.

For the second trajectory Γ_P, we consider its global behavior in the second quadrant, which we call G. It is known that any trajectory in G must begin and end either at z_1, or at infinity, or on the boundary of G. It is also known that there can be no closed Jordan curve consisting of trajectories, since there are no poles of the quadratic differential in G. For these properties of trajectories of quadratic differentials, see [34, Chapter 8] and also [4]. So the second critical trajectory that starts at z_1, cannot end at z_1, and it should end in G either on the negative real axis, or on the positive imaginary axis, or at infinity. If Γ_P meets the positive imaginary axis, then together with its mirror image in the right half-plane, it would be a closed Jordan curve not enclosing a pole. This is impossible. Next assume that Γ_P ends at infinity. At infinity, we have $-(\psi_Q - \psi_P)^2(\infty) = -1$. This means that the quadratic differential has a pole of order 4 at infinity, see [41], and all trajectories that extend to infinity arrive there with a vertical tangent. Assume that two critical trajectories extend to infinity in G. Then these trajectories are the boundary of a region G' in G. Any trajectory in G' begins and ends at infinity with a vertical tangent. However, according to [41, Theorem 7.4] a pole of order 4 has a neighborhood, so that any closed trajectory lying entirely in that neighborhood begins and ends at the pole, but from opposite directions. This is a contradiction, since there will be trajectories in G' that are arbitrarily close to infinity. This contradiction shows that there can be at most one critical trajectory that extends to infinity, namely $\Gamma_{E,1}$. Hence Γ_P can only meet the negative real axis. Then, by symmetry, its mirror image in the lower half-plane will be its continuation to z_2, so that Γ_P connects z_1 with z_2 in the left half-plane. By symmetry with respect to the imaginary axis, the third trajectory Γ_Q connects z_1 with z_2 in the right half-plane. Finally, by symmetry with respect to the real axis, we also find the trajectory $\Gamma_{E,2}$ that emanates from z_2 and extends to infinity. This completes the proof of Proposition 2.4.

3.2. Precise structure of the Riemann surface

Proof of Proposition 2.3. Assertion (a) is clear since ψ_P and ψ_Q are given explicitly by (2.11), where the square root is defined outside the cut Γ_P. Assertion (b) follows from (3.2) and (3.3). Finally, in the proof of Proposition 2.4, we have shown the existence of the curve Γ_P lying in the left half-plane and satisfying relation (2.13). This establishes assertion (c) of Proposition 2.3, which is proved completely.

4. Measures and functions associated with the Riemann surface

In this section we study the measures μ_P, μ_Q, μ_E, and the functions φ_P and g_P. These measures and functions that are associated with the Riemann surface satisfy several relations that will be used in the transformations of the Riemann–Hilbert problem that follow in later sections. We also prove Theorem 2.6 and Lemma 2.9 in this section.

4.1. Properties of the measures μ_P, μ_Q, and μ_E

We start with a lemma.

Lemma 4.1. *We have*

$$\frac{1}{\pi i}\int_{\Gamma_P}(\psi_Q-\psi_P)_+(s)\,ds = 1. \tag{4.1}$$

Proof. Let γ be a closed contour on the sheet $\mathcal{R}_P$ going around Γ_P once in the positive direction. Then the residue theorem for the exterior of γ gives

$$\frac{1}{2\pi i}\int_{\gamma}\psi_P(s)\,ds = \frac{1}{2},$$

because ψ_P is analytic outside γ and we have (3.2). If we shrink γ to Γ_P, then the integral becomes

$$\frac{1}{\pi i}\int_{\Gamma_P}((\psi_P)_-(s)-(\psi_P)_+(s))\,ds=1.$$

Taking into account that $(\psi_P)_-=(\psi_Q)_+$, we obtain (4.1).

Now we can prove Theorem 2.6.

Proof of Theorem 2.6. The curve Γ_P is such that for $z\in\Gamma_P$, the integral $\frac{1}{\pi i}\int_{z_1}^{z}(\psi_Q-\psi_P)_+(s)\,ds$ is real, see Proposition 2.3. For $z=z_1$, it has the value 0, and for $z=z_2$ it has the value 1 by (4.1). Let $z=z(t)$, $t\in[0,T]$, be arclength parametrization of Γ_P. The derivative of

$$t\mapsto\frac{1}{\pi i}\int_{z_1}^{z(t)}(\psi_Q-\psi_P)_+(s)\,ds \tag{4.2}$$

is equal to $\frac{1}{\pi i}\left(\psi_Q(z(t))-\psi_P(z(t))\right)z'(t)$ and this is different from 0 for $t\in(0,T)$. Thus (4.2) is strictly increasing from 0 for $t=0$ to 1 for $t=T$. This immediately implies that μ_P defined by (2.16) is a probability measure on Γ_P. By symmetry, μ_Q is a probability measure on Γ_Q.

For μ_E we observe that (2.17) defines a real measure on Γ_E, since, by Proposition 2.3, $\frac{1}{\pi i}\int_{z_1}^{z}(\psi_Q-\psi_P)(s)\,ds$ is real for $z\in\Gamma_{E,1}\cup\Gamma_{E,2}$. Using an argument based on arclength parametrization, similar to the one above, we find that on each part $\Gamma_{E,j}$, $j=1,2$, the measure μ_E is either positive or negative. Since $\psi_Q(s)-\psi_P(s)=1+\mathcal{O}(1/s^2)$ as $s\to\infty$ (see (3.2)–(3.3)), we have for $z\in\Gamma_{E,1}$,

$$\frac{1}{\pi i}\int_{z_1}^{z}(\psi_Q-\psi_P)(s)\,ds=\frac{1}{\pi i}(z-z_1)+\mathcal{O}(1)\qquad\text{as } z\to\infty,\quad z\in\Gamma_{E,1}.$$

Since $\operatorname{Im}(z-z_1)\to+\infty$ as $z\to\infty$ along $\Gamma_{E,1}$, the integral is positive as $z\to\infty$ along $\Gamma_{E,1}$. As the measure μ_E is of constant sign on $\Gamma_{E,1}$, we may thus deduce that it is positive everywhere on $\Gamma_{E,1}$. The reasoning for $\Gamma_{E,2}$ is similar.

4.2. Properties of g_P and φ_P

The function g_P was defined in (2.20). This is a multi-valued function, depending on the choice of the branch of the logarithm $\log(z-s)$, which we assume depends on $s\in\Gamma$ in a continuous way. Since μ_P is a probability measure, the g-function is defined modulo $2\pi i$.

Lemma 4.2. *For the derivative of the function g_P we have*

$$g_P'(z)=2\psi_P(z)+1,\qquad z\in\mathbb{C}\setminus\Gamma_P, \tag{4.3}$$

Proof. The derivative of g_P is easily obtained as

$$g_P'(z)=\frac{1}{\pi i}\int_{\Gamma_P}\frac{1}{z-s}(\psi_Q-\psi_P)_+(s)\,ds.$$

If γ is a closed contour going around Γ_P on $\mathcal{R}_P$ in the positive direction but with z outside γ, then (since $(\psi_Q)_+ = (\psi_P)_-$)

$$g'_P(z) = \frac{1}{\pi i} \oint_\gamma \frac{\psi_P(s)}{z-s}\, ds.$$

The integral over γ can be calculated with the residue theorem for the exterior of γ, for which there is a residue at z and at ∞ given by (3.2). This proves (4.3).

A useful explicit expression of g_P in terms of the mapping function ψ_P is given in the next lemma.

Lemma 4.3. *The function g_P has the following explicit expression in terms of the mapping functions ψ_P,*

$$g_P(z) = 2z\left(\psi_P(z) + \frac{1}{2}\right) - \log\left(\psi_P^2(z) - \frac{1}{4}\right) - 1 - \log(-2), \quad \text{for } z \in \mathbb{C} \setminus \Gamma_P. \tag{4.4}$$

Proof. We let $z \in \mathbb{C} \setminus \Gamma_P$ and put $w = \psi_P(z)$. Taking a derivative of (2.9), we find, since $w'(z) = \frac{1}{z'(w)}$,

$$2 = -\left(\frac{1}{(w-1/2)^2} + \frac{1}{(w+1/2)^2}\right) w'$$

Thus

$$\begin{aligned} 2\psi_P(z) + 1 &= -\left(w + \frac{1}{2}\right)\left(+\frac{1}{(w-1/2)^2} + \frac{1}{(w+1/2)^2}\right) w' \\ &= -\left(\frac{1}{w+1/2} + \frac{1}{w-1/2} + \frac{1}{(w-1/2)^2}\right) w' \\ &= \frac{d}{dz}\left[-\log(w^2 - 1/4) + \frac{1}{w-1/2}\right]. \end{aligned}$$

By (4.3), we then see that

$$g_P(z) = -\log(w^2 - 1/4) + \frac{1}{w-1/2} + C$$

for some constant C. The constant can be determined from the behavior for $z \to \infty$, since $g_P(z) = \log z + \mathcal{O}(1/z)$, and $w = \psi_P(z) = -\frac{1}{2} + \frac{1}{2z} + \mathcal{O}(1/z^2)$. The result is that $C = 1 - \log(-2)$. Using (2.9) we then find (4.4).

We recall that the function φ_P was introduced in (2.21). The next lemma connects φ_P with g_P, and gives jump properties of g_P across Γ_P. It will be frequently used in what follows. Throughout the rest of the paper we use ℓ to denote the constant

$$\ell = -\frac{i\pi}{2}. \tag{4.5}$$

Lemma 4.4. *For $z \in \mathbb{C} \setminus \Gamma_P$, we have*

$$g_P(z) = \log z + z - \varphi_P(z) + \ell, \tag{4.6}$$

On the contours we have

$$g_{P+}(z) + g_{P-}(z) = 2\,\log z + 2z + 2\ell, \qquad z \in \Gamma_P, \tag{4.7}$$

and

$$\begin{aligned} g_{P+}(z) - g_{P-}(z) &= -\varphi_{P+}(z) + \varphi_{P-}(z) \\ &= -2\varphi_{P+}(z) = 2\varphi_{P-}(z), \quad z \in \Gamma_P. \end{aligned} \tag{4.8}$$

Proof. Integrating (4.3) from z_1 to z over some path in $\mathbb{C} \setminus \Gamma_P$, we get

$$g_P(z) - g_P(z_1) = 2\int_{z_1}^{z} \psi_P(s)\,ds + (z - z_1), \tag{4.9}$$

$$= -2\varphi_P(z) + 2\int_{z_1}^{z} \psi_Q(s)\,ds + (z - z_1), \tag{4.10}$$

$$= -2\varphi_P(z) + 2\int_{z_1}^{z} \frac{ds}{s} - 2\int_{z_1}^{z} \psi_P(s)\,ds + (z - z_1). \tag{4.11}$$

Hence

$$g_P(z) = \log z + z - \varphi_P(z) + (g_P(z_1) - \log z_1 - z_1),$$

so that (4.6) holds with constant

$$\ell = g_P(z_1) - \log z_1 - z_1.$$

Using the explicit expressions (4.4) for g_P we are able to show that ℓ is equal to (4.5).

Next, we use Lemma 4.2 to find

$$g_{P+}(z) + g_{P-}(z) = 2\int_{z_1}^{z} (\psi_{P+} + \psi_{P-})(s)\,ds + 2(z - z_1) + 2g(z_1), \quad z \in \Gamma_P.$$

On Γ_P we have $\psi_{P-}(s) = \psi_{Q+}(s)$, so that

$$g_{P+}(z) + g_{P-}(z) = 2\int_{z_1}^{z} (\psi_P + \psi_Q)_+(s)\,ds + 2z - 2z_1 + 2i, \quad z \in \Gamma_P.$$

Since $\psi_P(s) + \psi_Q(s) = \frac{1}{s}$, we obtain

$$g_{P+}(z) + g_{P-}(z) = 2\log z - 2\,\log z_1 + 2z = 2\,\log z + 2z + 2\ell, \quad z \in \Gamma_P.$$

This proves (4.7). Finally, if we take (4.6) on the $+$ and $-$sides of Γ_P and subtract, we get

$$g_{P+}(z) - g_{P-}(z) = -\varphi_{P+}(z) + \varphi_{P-}(z), \quad z \in \Gamma_P.$$

This gives (4.8), since $\varphi_{P+}(z) = -\varphi_{P-}(z)$.

Since $g_P(z) = \log z + \mathcal{O}\,(1/z)$ as $z \to \infty$, we get from (4.6) that

$$\varphi_P(z) = z + \ell + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } z \to \infty. \tag{4.12}$$

We also see from (4.6) that φ_P is a multivalued function, which is defined modulo $2\pi i$, since g_P is defined modulo $2\pi i$.

Proof of Lemma 2.9. By Proposition 2.3 and the definition of φ_P, we have that $\operatorname{Re}\varphi_P = 0$ on Γ_P, Γ_Q, $\Gamma_{E,1}$, and $\Gamma_{E,2}$.

We know that $\operatorname{Re}\varphi_P$ is a harmonic function in $\mathbb{C} \setminus (\Gamma_P \cup \{0\})$. Since $\psi_Q(z) \sim 1/z$ as $z \to 0$, it easily follows from (2.21) that $\operatorname{Re}\varphi_P(z) \to -\infty$ as $z \to 0$. Then by the maximum principle for harmonic functions we get that $\operatorname{Re}\varphi_P < 0$ on D_0.

As $z \to \infty$, we have (4.12). On the unbounded curves $\Gamma_{E,1}$ and $\Gamma_{E,2}$ we have $\operatorname{Re}\varphi_P = 0$. As $z \to \infty$ in the unbounded domain $D_{\infty,P}$ we have $\limsup \operatorname{Re}(z + \ell) \leqslant 0$, so that $\limsup \operatorname{Re}\varphi_P(z) \leqslant 0$ by (4.12). Again it follows by the maximum principle for harmonic functions that $\operatorname{Re}\varphi_P < 0$ on $D_{\infty,P}$.

For the remaining domain $D_{\infty,Q}$, we have that $\operatorname{Re}\varphi_P$ is harmonic, with $\liminf \operatorname{Re}\varphi_P(z) \geqslant 0$ as $z \to \infty$ with $z \in D_{\infty,Q}$ by (4.12). Thus again by the maximum principle, $\operatorname{Re}\varphi_P > 0$ on $D_{\infty,Q}$. This completes the proof of Lemma 2.9.

5. The Riemann–Hilbert problem and the first two transformations

Throughout Sections 5 and 6, we will restrict the analysis to *normal indices* with respect to the given scheme $B := B^{(2n)} = \{z_i^{(2n)}\}_{i=0}^{2n}$ of interpolation points, namely indices $n \in \Lambda$ where

$$\Lambda = \{n \in \mathbb{N}, \quad Q_n \text{ satisfying (1.3) has exact degree } n\}. \tag{5.1}$$

Moreover, we assume that *the scheme B of interpolation points is such that the set Λ is infinite*.

Remark. Let $n \in \Lambda$ and let $p_{n+1,n-1}$ and $q_{n+1,n-1}$ be the rational interpolants of type $(n+1, n-1)$ to e^z satisfying

$$p_{n+1,n-1}(z) + q_{n+1,n-1}(z)e^z = \mathcal{O}\left(\omega_{2n+1}(z)\right), \quad \text{where}$$
$$\omega_{2n+1}(z) = \prod_{i=0}^{2n}\left(z - z_i^{(2n)}\right).$$

Then $p_{n+1,n-1}$ is of exact degree $n+1$. Indeed, assume that $\deg p_{n+1,n-1} < n+1$. Then the pair $(p_{n+1,n-1}, q_{n+1,n-1})$ would be a pair of type $(n, n-1)$ solving the rational interpolation problem of type (n, n), which is impossible since $n \in \Lambda$.

Our asymptotic analysis is based on the Riemann–Hilbert problem for Y formulated in the introduction, see (1.4) and (1.5). In this section we prove that, when n belongs to Λ, the Riemann–Hilbert problem has a unique solution and that the solution is given in terms of the polynomials P_n, Q_n, and the remainder E_n. We also do the first two transformations of the Riemann–Hilbert problem, which consist of a normalization of the problem at infinity, and a deformation of contours.

5.1. The Riemann–Hilbert problem

We show that the Riemann–Hilbert problem for Y and $n \in \Lambda$ has a solution in terms of the rational interpolants.

Theorem 5.1. *Let $n \in \Lambda$ and P_n, Q_n, E_n, and Ω_n be as above. We assume that n is large enough so that all roots of Ω_n are inside the contour Γ. Then the solution of the Riemann–Hilbert problem for Y (see the introduction) is unique and is given by*

$$Y(z) = \begin{pmatrix} p_{n+1,n-1}(2nz) & \Omega_n^{-1}(z) q_{n+1,n-1}(2nz) \\ P_n(z) & \Omega_n^{-1}(z) Q_n(z) \end{pmatrix}, \tag{5.2}$$

for z outside Γ, and

$$Y(z) = \begin{pmatrix} p_{n+1,n-1}(2nz) & \Omega_n^{-1}(z) e^{-nz} e_{n+1,n-1}(2nz) \\ P_n(z) & \Omega_n^{-1}(z) e^{-nz} E_n(z) \end{pmatrix}, \tag{5.3}$$

for z inside Γ. In the first rows of (5.2) and (5.3) we use the rational interpolants of indices $n+1, n-1$ normalized so that $p_{n+1,n-1}(2nz)$ is a monic polynomial of degree $n+1$, which, in view of the previous remark, is possible. In the second row we use the rational interpolants of indices n, n normalized so that $Q_n(z) = q_{n,n}(2nz)$ is a monic polynomial of degree n.

Proof. The given Y is analytic inside and outside the contour Γ. This is clear from (5.2) and (5.3), except perhaps for the second column of (5.3) which has possible singularities at the roots of Ω_n. However, the singularities are removable since $e_{n_1,n_2}(z) = \mathcal{O}\left(\omega_{n_1+n_2+1}(z)\right)$. Next, the normalizations such that $p_{n+1,n-1}(2nz)$ and $Q_n(z)$ are monic polynomials of exact degree $n+1$ and n respectively can always be performed since $n \in \Lambda$. From this choice of normalizations, we see that the asymptotic condition (1.5) is satisfied.

The jump condition can easily be checked. For the entries in the first column it reads

$$(Y_{11})_+ = (Y_{11})_-, \quad (Y_{21})_+ = (Y_{21})_-,$$

which is indeed so since Y_{11} and Y_{21} are both polynomials. We also have

$$\begin{aligned} (Y_{12})_+(z) &= \Omega_n^{-1}(z) e^{-nz} e_{n+1,n-1}(2nz) \\ &= \Omega_n^{-1}(z) \left(p_{n+1,n-1}(2nz) e^{-2nz} + q_{n+1,n-1}(2nz) \right) \\ &= \Omega_n^{-1}(z) e^{-2nz} (Y_{11})_-(z) + (Y_{12})_-(z) \end{aligned}$$

and this is the jump condition (1.4) for the second entry in the first row. The last entry is handled in the same way.

To prove uniqueness, we assume that $\widetilde{Y}$ is another solution of the Riemann–Hilbert problem. First observe that $\det Y$ is a scalar function which is analytic in $\mathbb{C} \setminus \Gamma$. Because of (1.4), we have that $(\det Y)_+(z) = (\det Y)_-(z)$ for $z \in \Gamma$, so that $\det Y$ has no jump, making $\det Y$ an entire function. For large z we have $\det Y(z) = 1 + \mathcal{O}(1/z)$ by (1.5), hence by Liouville's theorem $\det Y = 1$ everywhere. We can therefore consider $\widetilde{Y} Y^{-1}$, which is analytic in $\mathbb{C} \setminus \Gamma$. There is no jump on Γ since $(\widetilde{Y} Y^{-1})_+(z) = (\widetilde{Y} Y^{-1})_-(z)$ for every $z \in \Gamma$, hence $\widetilde{Y} Y^{-1}$ is entire (i.e., each entry is an entire function). For large z we have $\widetilde{Y}Y^{-1}(z) = I + \mathcal{O}(1/z)$, hence Liouville's theorem implies that $\widetilde{Y}Y^{-1}(z) = I$ for every z, and hence $\widetilde{Y}(z) = Y(z)$.

5.2. *First transformation*

We will use the function g_P, and the constant $\ell = -\frac{i\pi}{2}$ from Section 4.3 to transform the Riemann–Hilbert problem for Y to a Riemann–Hilbert problem for U, given by

$$U(z) = L^{-n-1}Y(z)\begin{pmatrix} e^{-(n+1)g_P(z)} & 0 \\ 0 & e^{(n+1)g_P(z)} \end{pmatrix} L^{n+1}, \tag{5.4}$$

where L is the constant diagonal matrix

$$L = \begin{pmatrix} e^{\ell} & 0 \\ 0 & e^{-\ell} \end{pmatrix}. \tag{5.5}$$

For the contour Γ we take $\Gamma = \Gamma_P \cup \Gamma_R$, where Γ_R is a contour connecting z_2 to z_1 and lying in $D_{\infty,Q}$. Then $\operatorname{Re}\varphi_P > 0$ on Γ_R by Lemma 2.9.

We note that U is analytic on $\mathbb{C}\setminus\Gamma$, since $e^{g_P(z)}$ is analytic and single-valued on $\mathbb{C}\setminus\Gamma_P$ and $\Gamma_P \subset \Gamma$.

Since $g_P(z) = \log z + \mathcal{O}(1/z)$ as $z \to \infty$, we have $e^{(n+1)g_P(z)} = z^{n+1}[1 + \mathcal{O}(1/z)]$ as $z \to \infty$. Hence

$$U(z) = I + \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty. \tag{5.6}$$

So U is normalized at infinity.

The jump relation for U needs to be worked out on the two pieces of the contour $\Gamma = \Gamma_P \cup \Gamma_R$. For $z \in \Gamma_P$ we have

$$U_+(z) = U_-(z)\begin{pmatrix} e^{-(n+1)[g_{P+}(z)-g_{P-}(z)]} & ze^{2z}s_n^{-1}(z)e^{(n+1)[-2\log z-2z+g_{P+}(z)+g_{P-}(z)-2\ell]} \\ 0 & e^{(n+1)[g_{P+}(z)-g_{P-}(z)]} \end{pmatrix}. \tag{5.7}$$

Taking into account Lemma 4.4, we can simplify the jump (5.7) to

$$U_+(z) = U_-(z)\begin{pmatrix} e^{2(n+1)\varphi_{P+}(z)} & ze^{2z}s_n^{-1}(z) \\ 0 & e^{2(n+1)\varphi_{P-}(z)} \end{pmatrix}, \quad z \in \Gamma_P.$$

On the part Γ_R we have

$$U_+(z) = U_-(z)\begin{pmatrix} 1 & ze^{2z}s_n^{-1}(z)e^{(n+1)[-2\log z-2z+2g_P(z)-2\ell]} \\ 0 & 1 \end{pmatrix}. \tag{5.8}$$

If we use Lemma 4.4 then (5.8) can be re-written as

$$U_+(z) = U_-(z)\begin{pmatrix} 1 & ze^{2z}s_n^{-1}(z)e^{-2(n+1)\varphi_P(z)} \\ 0 & 1 \end{pmatrix}, \qquad z \in \Gamma_R.$$

Summarizing, we have the following Riemann–Hilbert problem for U

1. U is analytic on $\mathbb{C}\setminus\Gamma$.

2. U satisfies the following jump relations

$$U_+(z) = U_-(z) \begin{pmatrix} e^{2(n+1)\varphi_{P+}(z)} & ze^{2z}s_n^{-1}(z) \\ 0 & e^{2(n+1)\varphi_{P-}(z)} \end{pmatrix}, \qquad z \in \Gamma_P, \tag{5.9}$$

$$U_+(z) = U_-(z) \begin{pmatrix} 1 & ze^{2z}s_n^{-1}(z)e^{-2(n+1)\varphi_P(z)} \\ 0 & 1 \end{pmatrix}, \qquad z \in \Gamma_R. \tag{5.10}$$

3. $U(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

The contour Γ_R is in the region where Re φ_P is positive. The jump matrix in (5.10) for U on the contour Γ_R is then the identity matrix I plus a matrix with entries that tend to zero exponentially fast as $n \to \infty$.

The function $\varphi_{P+} = -\varphi_{P-}$ is purely imaginary on Γ_P because of (2.13), so that the diagonal elements of the jump matrix on Γ_P are oscillatory.

5.3. Deformation of contours

The jump matrix in (5.9) can be written as a product of three matrices

$$\begin{aligned} &\begin{pmatrix} e^{2(n+1)\varphi_{P+}(z)} & ze^{2z}s_n^{-1}(z) \\ 0 & e^{2(n+1)\varphi_{P-}(z)} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ z^{-1}e^{-2z}s_n(z)e^{2(n+1)\varphi_{P-}(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & ze^{2z}s_n^{-1}(z) \\ -z^{-1}e^{-2z}s_n(z) & 0 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 1 & 0 \\ z^{-1}e^{-2z}s_n(z)e^{2(n+1)\varphi_{P+}(z)} & 1 \end{pmatrix}. \end{aligned} \tag{5.11}$$

Instead of jumping over Γ_P in one jump, we will make three smaller jumps, and rather than jumping over one contour, we jump over three contours, and each contour deals with one of the matrices in the product (5.11). We will open up a lens around Γ_P, which consists of two contours $\Gamma_{P-} \cup \Gamma_{P+}$ connecting z_1 and z_2, such that Γ_{P-} is on the minus side of Γ_P and Γ_{P+} is on the plus side of Γ_P, but still inside the region where Re $\varphi_P < 0$.

These contours are drawn in Fig. 5.

All together there are 4 contours, and they determine 4 regions in the plane. The second transformation $U \mapsto T$ will be defined in each of these regions separately. We define T as follows. We take

$$T(z) = U(z), \tag{5.12}$$

for z in the unbounded region, and in the middle region bounded by Γ_{P+} and Γ_R. In the two regions near Γ_P we put

$$T(z) = \begin{cases} U(z)V_{P-}(z) & \text{for } z \text{ in the region bounded by } \Gamma_{P-} \text{ and } \Gamma_P, \\ U(z)V_{P+}^{-1}(z) & \text{for } z \text{ in the region bounded by } \Gamma_P \text{ and } \Gamma_{P+}, \end{cases} \tag{5.13}$$

where

$$V_{P-}(z) = V_{P+}(z) = \begin{pmatrix} 1 & 0 \\ z^{-1}e^{-2z}s_n(z)e^{2(n+1)\varphi_P(z)} & 1 \end{pmatrix}. \tag{5.14}$$

Then we have the following Riemann–Hilbert problem for T.

1. T is analytic in each of the 4 regions,

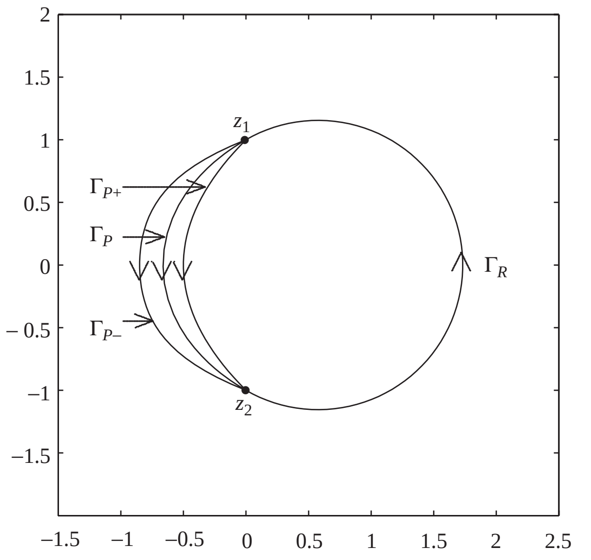

Fig. 5. Deformation of contours around Γ_P.

2. T has a jump on each of the 4 contours

$$T_+(z) = T_-(z)V_s(z), \qquad z \in \Gamma_s,$$

where s stand for any of the four symbols P, P^-, P^+, R. The matrices V_{P^-}, V_{P^+} have already been defined above. The other jump matrices are

$$V_P(z) = \begin{pmatrix} 0 & ze^{2z}s_n^{-1}(z) \\ -z^{-1}e^{-2z}s_n(z) & 0 \end{pmatrix}, \tag{5.15}$$

$$V_R(z) = \begin{pmatrix} 1 & ze^{2z}s_n^{-1}(z)e^{-2(n+1)\varphi_P(z)} \\ 0 & 1 \end{pmatrix}, \tag{5.16}$$

3. $T(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

Observe that all jumps, except for the jump on Γ_P, tend to the identity matrix exponentially fast as $n \to \infty$. Hence we expect that the dominating contribution is the jump V_P on Γ_P.

6. Construction of parametrices and final transformation

6.1. Parametrix for the exterior region

We will now solve a Riemann–Hilbert problem for a matrix valued function N on the contour Γ_P which, in view of what was said at the end of the previous section, is expected to describe the main contribution of the Riemann–Hilbert problem of T.

We look for $N : \mathbb{C} \setminus \Gamma_P \to \mathbb{C}^{2\times 2}$ satisfying

1. N is analytic in $\mathbb{C} \setminus \Gamma_P$.
2. N has jump on Γ_P given by

$$N_+(z) = N_-(z) \begin{pmatrix} 0 & ze^{2z}s_n^{-1}(z) \\ -z^{-1}e^{-2z}s_n(z) & 0 \end{pmatrix}, \qquad z \in \Gamma_P. \tag{6.1}$$

3. $N(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

Note that, since the jump in relation (6.1) involves the function s_n, the matrix N will depend on the degree n. Recall also that the functions $\widetilde{s}_n$ and $\widehat{s}_n$ were defined in (2.37).

Proposition 6.1. *A solution of the Riemann–Hilbert problem for N is given by*

$$N(z) = \begin{pmatrix} F_1(\psi_P(z)) & F_1(\psi_Q(z)) \\ F_2(\psi_P(z)) & F_2(\psi_Q(z)) \end{pmatrix}, \tag{6.2}$$

where

$$F_1(w) = -\frac{(w-1/2)G(w)}{\sqrt{2}\widehat{s}_n(-2)\sqrt{4w^2+1}}, \tag{6.3}$$

$$F_2(w) = \frac{\sqrt{2}\widetilde{s}_n(-2)(w+1/2)G(w)}{\sqrt{4w^2+1}}, \tag{6.4}$$

with $\sqrt{4w^2+1}$ defined and analytic in $\mathbb{C} \setminus \psi_{P+}(\Gamma_P)$, and such that it is positive for large positive w. The function G is defined by

$$G(w) = \begin{cases} 2(w-1/2)\widehat{s}_n(1/w)e^{-\frac{(w+1/2)}{(w-1/2)}} & \text{for } w \in \psi(\mathcal{R}_P), \\ \frac{2w}{(w+1/2)\widetilde{s}_n(-4w)}e^{-\frac{(w-1/2)}{(w+1/2)}} & \text{for } w \in \psi(\mathcal{R}_Q). \end{cases} \tag{6.5}$$

Proof. Let us consider the first row (N_{11}, N_{12}) of N. From (6.1) we get the following jumps on Γ_P

$$\begin{cases} (N_{11})_+(z) = -z^{-1}e^{-2z}s_n(z)(N_{12})_-(z), \\ (N_{12})_+(z) = ze^{2z}s_n^{-1}(z)(N_{11})_-(z), \end{cases} \qquad z \in \Gamma_P, \tag{6.6}$$

We can see N_{11} as a function on the sheet $\mathcal{R}_P$ of the Riemann surface $\mathcal{R}$ and N_{12} as a function on $\mathcal{R}_Q$. Then we transform the problem from $\mathcal{R}$ with the variable z, to the complex w-plane, via the mapping $\psi : \mathcal{R} \to \mathbb{C}$. The variables z and w are connected by (2.9). The images of the two sheets, the images of the branch points z_1, z_2, and the image of the cut Γ_P are shown in Fig. 2.

Note that the images of Γ_P under the mappings ψ_{P+} and ψ_{P-} (positive and negative boundary values of ψ_P on Γ_P) give two arcs from w_1 to w_2. They are oriented as shown in Fig. 2. The orientation corresponds to the orientation of Γ_P. Together the arcs make up a simple closed loop around $-1/2$.

Now we transplant the (as yet unknown) functions N_{11} and N_{12} from the Riemann surface to the w-plane, by defining F_1 as follows:

$$F_1(w) = \begin{cases} N_{11}\left(\frac{w}{w^2-1/4}\right), & w \in \psi(\mathcal{R}_P), \\ N_{12}\left(\frac{w}{w^2-1/4}\right), & w \in \psi(\mathcal{R}_Q). \end{cases} \tag{6.7}$$

Then F_1 is analytic in $\mathbb{C} \setminus \psi_{P\pm}(\Gamma_P)$. The jumps that F_1 should satisfy can be determined from (6.6) and are given by

$$\begin{cases} F_{1+}(w) = ze^{2z}s_n^{-1}(z)F_{1-}(w), & w \in \psi_{P-}(\Gamma_P), \\ F_{1+}(w) = -z^{-1}e^{-2z}s_n(z)F_{1-}(w), & w \in \psi_{P+}(\Gamma_P), \end{cases} \tag{6.8}$$

where $z = z(w) = \frac{w}{w^2-1/4}$.

The asymptotic condition on N implies that $N_{11}(z) \to 1$, $N_{12}(z) \to 0$, as $z \to \infty$. For F_1, this means that

$$F_1(-1/2) = 1, \qquad F_1(1/2) = 0. \tag{6.9}$$

We also want $F_1(w)$ to have a finite limit as $w \to \infty$, since $w = \infty$ corresponds to $z = 0$ on the Q-sheet.

We now seek F_1 in the form

$$F_1(w) = -\frac{(w-1/2)G(w)}{\sqrt{2\widehat{s}_n(-2)}\sqrt{4w^2+1}}. \tag{6.10}$$

Then G should be analytic in $\mathbb{C} \setminus \psi_{P\pm}(\Gamma_P)$ with jumps

$$\begin{cases} G_+(w) = ze^{2z}s_n^{-1}(z)G_-(w), & w \in \psi_{P-}\Gamma_P), \\ G_+(w) = z^{-1}e^{-2z}s_n(z)G_-(w), & w \in \psi_{P+}(\Gamma_P), \end{cases} \tag{6.11}$$

with $z = z(w)$. The normalization for G is

$$G(-1/2) = -2\widehat{s}_n(-2). \tag{6.12}$$

Taking logarithms in (6.11) and using the well-known Plemelj formula, one reconstructs G as given by (6.5), which indeed satisfies (6.11) and (6.12). Then by (6.10) it follows that F_1 has the correct jumps (6.8) and normalization (6.9). Then from (6.7) we recover N_{11} and N_{12} in terms of F_1 by

$$N_{11}(z) = F_1(\psi_P(z)), \quad N_{12}(z) = F_1(\psi_Q(z)).$$

Then the jump (6.6) is satisfied, and in addition the normalization at infinity is correct. So we have found the first row of N.

The proof for the second row is similar. The only difference is that we have a different normalization at infinity, which leads to the construction of a function F_2 that satisfies the same jump (6.8) as F_1, but is normalized by

$$F_2(-1/2) = 0, \qquad F_2(1/2) = 1.$$

Similar calculations then lead to the formula (6.4) with the same function G.

We remark that the entries of N have fourth root singularities at the two branch points z_1 and z_2. More precisely,

Lemma 6.2. *For each given n sufficiently large, the entries of N behave as follows near the branch points. As $z \to z_j$ with $j = 1, 2$, we have*

$$\begin{cases} N_{k1}(z) = \mathcal{O}\left(|z-z_j|^{-1/4}\right), \\ N_{k2}(z) = \mathcal{O}\left(|z-z_j|^{-1/4}\right), \end{cases} \qquad k = 1, 2. \tag{6.13}$$

Proof. Since w_1 is a non-degenerate critical point of the mapping $z = z(w)$, we have for the inverse $w = \psi_P(z)$ as $z \to z_1 = z(w_1)$,

$$\psi_P(z) = w_1 + c(z - z_1)^{1/2} + \mathcal{O}\ (z - z_1) \tag{6.14}$$

where c is a non-zero constant. Since w_1 is a simple root of $4w^2 + 1$, it then follows that

$$\sqrt{4\psi_P^2(z) + 1} = c_2(z - z_1)^{1/4} + \mathcal{O}\ \left(|z - z_1|^{3/4}\right), \qquad z \to z_1, \tag{6.15}$$

with $c_2 \neq 0$. Since the numerators of F_1, F_2 as given by (6.3) and (6.4), do not vanish for $w = w_1$, we find

$$N_{k1}(z) = F_k(\psi_P(z)) = \mathcal{O}\ \left(|z - z_1|^{-1/4}\right), \qquad k = 1, 2,$$

as $z \to z_1$. In a similar way we find that $N_{k2}(z) = \mathcal{O}\ \left(|z - z_1|^{-1/4}\right)$ as $z \to z_1$. This proves (6.13) for $j = 1$.

The behavior near the other branch point z_2 follows in a similar way.

Remark. It will be useful to have another representation for the entries in the second row of N. They are

$$\begin{aligned} N_{21}(z) &= -\frac{\sqrt{2}\widetilde{s}_n(-2)\widehat{s}_n(1/\psi_P(z))e^{-g_P(z)}}{\sqrt{4\psi_P^2(z) + 1}}, \\ N_{22}(z) &= \frac{\sqrt{2}\widetilde{s}_n(-2)e^{g_P(z)}}{z\widetilde{s}_n(-4\psi_Q(z))\sqrt{4\psi_Q^2(z) + 1}}. \end{aligned} \tag{6.16}$$

It may be checked directly that these functions have the right asymptotics as $z \to \infty$, and satisfy the correct jump relations on Γ_P. They also satisfy the $\mathcal{O}$ -conditions of Lemma 6.2.

The Riemann–Hilbert problem for T is now very close to the Riemann–Hilbert problem for N because the jumps for T and N on the contour Γ_P are the same and the jumps for T on the other contours tend to the identity matrix as $n \to \infty$, uniformly away from the branch points. So we expect that T behaves like N as $n \to \infty$ away from the branch points. However, in order to justify this, a more detailed analysis of the Riemann–Hilbert problem near the branch points is needed.

6.2. Parametrices near the branch points

Before starting the construction of these parametrices, we state some relations and estimates satisfied by the functions s_n, that were defined in (2.22). Recall that the interpolation points $z_i^{(2n)}$ are subject to the growth condition (2.1).

Lemma 6.3. *For any z_1, z_2 and z_3 in $\overline{\mathbb{C}} \setminus \{0\}$ such that $z_1^{-1} + z_2^{-1} = z_3^{-1}$, we have*

$$s_n(z_1)s_n(z_2) = s_n(z_3)\left(1 + \mathcal{O}\ \left(\frac{\log^2 n}{n}\right)\right), \qquad n \to \infty. \tag{6.17}$$

In particular, with $z_1 = z$, $z_2 = -z$ *and* $z_3 = \infty$, *we get*

$$s_n(z)s_n(-z) = 1 + \mathcal{O}\left(\frac{\log^2 n}{n}\right), \qquad n \to \infty. \tag{6.18}$$

We also have

$$s_n(1/\psi_P(z))s_n(1/\psi_Q(z)) = s_n(z)\left(1 + \mathcal{O}\left(\frac{\log^2 n}{n}\right)\right), \qquad n \to \infty. \tag{6.19}$$

Moreover,

$$s_n(z) = \mathcal{O}\left(n^{\frac{1-\alpha}{2|z|}}\right), \qquad s_n^{-1}(z) = \mathcal{O}\left(n^{\frac{1-\alpha}{2|z|}}\right), \qquad n \to \infty \tag{6.20}$$

and, for any z *in* $\overline{\mathbb{C}} \setminus \{0\}$,

$$|s_n(z) - 1| \leqslant (1 + \rho_n/(2n|z|))\left(\exp(\rho_n/|z|) - 1\right), \tag{6.21}$$

where ρ_n *has been defined in* (2.1) *as the radius of a disk* $\mathcal{D}_n$ *that contains all the points of* $B^{(2n)}$.

Proof. In order to prove the first estimate, we group together the factors in $s_n(z_1)$ and $s_n(z_2)$ corresponding to the same interpolation point $z_i^{(2n)}$. Then, recalling (2.1), we notice that, as $n \to \infty$,

$$\left(1 - \frac{z_i^{(2n)}}{2nz_1}\right)\left(1 - \frac{z_i^{(2n)}}{2nz_2}\right) = \left(1 - \frac{z_i^{(2n)}}{2nz_3}\right)\left(1 + \mathcal{O}\left(\frac{\log^2 n}{n^2}\right)\right).$$

Since, as $n \to \infty$,

$$\left(1 + \mathcal{O}\left(\frac{\log^2 n}{n^2}\right)\right)^{2n+1} = 1 + \mathcal{O}\left(\frac{\log^2 n}{n}\right),$$

equality (6.17) is proved. The number $\psi_P(z)$ and $\psi_Q(z)$ are the two roots of (2.10), hence

$$\psi_P(z) + \psi_Q(z) = \frac{1}{z},$$

so that (6.19) is a consequence of (6.17). The estimates (6.20) follow from the inequalities

$$\prod_{i=0}^{2n}\left(1 - \frac{|z_i^{(2n)}|}{2n|z|}\right) \leqslant |s_n(z)| \leqslant \prod_{i=0}^{2n}\left(1 + \frac{|z_i^{(2n)}|}{2n|z|}\right),$$

the estimates

$$e^x e^{-\frac{x^2}{n+x}} \leqslant \left(1 + \frac{x}{n}\right)^n \leqslant e^x, \qquad -n \leqslant x,$$

and (2.1). The inequality (6.21) follows also easily from the definition of s_n, (2.1), and the following elementary inequality: If $u_1, \ldots, u_N$ are complex numbers, then

$$\left|\prod_{n=1}^{N}(1 + u_n) - 1\right| \leqslant \prod_{n=1}^{N}(1 + |u_n|) - 1.$$

The construction of the local parametrices resembles the construction described in Section 6.2 of [27]. The parametrices consist of two new contours Γ_1 and Γ_2 which are small circles of radius $\delta_n = \mathcal{O}(1/\rho_n^2)$ as n tends to infinity, centered at the two branch points. Note that if α in (2.1) is different from 1, then δ_n depends on n and the contours Γ_1 and Γ_2 shrink to the branch points as n becomes large. Inside each of these contours the Riemann–Hilbert problem for T is solved exactly.

Zooming in near the branch point z_1 gives a Riemann–Hilbert problem with five contours $\Gamma_{P^-}, \Gamma_P, \Gamma_{P^+}, \Gamma_R, \Gamma_1$. The jumps on these contours are

$$V_{P^-}(z) = V_{P^+}(z) = \begin{pmatrix} 1 & 0 \\ z^{-1}e^{-2z}s_n(z)e^{2(n+1)\varphi_P(z)} & 1 \end{pmatrix}$$
$$V_P = \begin{pmatrix} 0 & ze^{2z}s_n^{-1}(z) \\ -z^{-1}e^{-2z}s_n(z) & 0 \end{pmatrix}$$
$$V_R(z) = \begin{pmatrix} 1 & ze^{2z}s_n^{-1}(z)e^{-2(n+1)\varphi_P(z)} \\ 0 & 1 \end{pmatrix}.$$

We look for a 2×2 matrix valued function $M^{(1)}$ defined within the disk Δ_1 surrounded by Γ_1, such that

1. $M^{(1)}$ is analytic in $\Delta_1 \setminus (\Gamma_P \cup \Gamma_{P^-} \cup \Gamma_{P^+} \cup \Gamma_R)$,
2. $M^{(1)}$ has the jumps
$$M_+^{(1)}(z) = M_-^{(1)}(z)V_s(z), \qquad z \in \Gamma_s,$$
where s stands for any of the symbols P, P^-, P^+, and R.
3. On Γ_1 we have that $M^{(1)}$ matches N in the sense that
$$M^{(1)}(z) = \left(I + \mathcal{O}\left(\frac{1}{n^{(\alpha+1)/2}}\right)\right)N(z) \tag{6.22}$$
uniformly for $z \in \Gamma_1$.

A local parametrix can be built which uses the conformal mapping f_1 from Δ_1 onto a convex neighborhood of 0 defined by

$$\varphi_P(z) = \frac{2}{3}\,[f_1(z)]^{3/2}.$$

It maps Γ_P onto a part of the negative real axis. Moreover, freedom is left, which allows one choosing Γ_R so that it is mapped to a part of the positive real line, and Γ_{P^-} and Γ_{P^+} so that they are mapped onto rays in the complex s-plane. We denote the images of Γ_P, Γ_{P^-}, Γ_{P^+}, and Γ_R, by Σ_P, Σ_{P^-}, Σ_{P^+} and Σ_R. These contours are shown in Fig. 6. On these contours we use the constant jump matrices

$$\widehat{V}_{P^-} = \widehat{V}_{P^+} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

on Σ_{P^-} and Σ_{P^+},

$$\widehat{V}_P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

on Σ_P, and

$$\widehat{V}_{U^P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

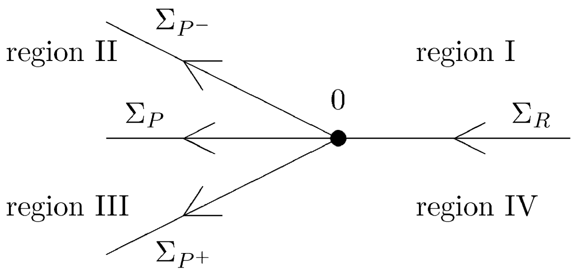

Fig. 6. The contours for $\Psi^{(1)}$

on Σ_R. This Riemann–Hilbert problem is well-known and its solution $\Psi^{(1)}$ is given in terms of the Airy function $\mathrm{Ai}(z)$ by

$$\begin{aligned}
\Psi^{(1)}(s) &= \begin{pmatrix} \mathrm{Ai}(s) & -\mathrm{Ai}(\omega_3^2 s) \\ \mathrm{Ai}'(s) & -\omega_3^2\mathrm{Ai}'(\omega_3^2 s) \end{pmatrix} \begin{pmatrix} e^{-i\pi/6} & 0 \\ 0 & e^{i\pi/6} \end{pmatrix}, \qquad s \in I, \\
\Psi^{(1)}(s) &= \begin{pmatrix} \mathrm{Ai}(s) & -\mathrm{Ai}(\omega_3^2 s) \\ \mathrm{Ai}'(s) & -\omega_3^2\mathrm{Ai}'(\omega_3^2 s) \end{pmatrix} \begin{pmatrix} e^{-i\pi/6} & 0 \\ 0 & e^{i\pi/6} \end{pmatrix} \widehat{V}_{P-}, \qquad s \in II, \\
\Psi^{(1)}(s) &= \begin{pmatrix} \mathrm{Ai}(s) & \omega_3^2\mathrm{Ai}(\omega_3 s) \\ \mathrm{Ai}'(s) & \mathrm{Ai}'(\omega_3 s) \end{pmatrix} \begin{pmatrix} e^{-i\pi/6} & 0 \\ 0 & e^{i\pi/6} \end{pmatrix} \widehat{V}_{P+}^{-1}, \qquad s \in III, \\
\Psi^{(1)}(s) &= \begin{pmatrix} \mathrm{Ai}(s) & \omega_3^2\mathrm{Ai}(\omega_3 s) \\ \mathrm{Ai}'(s) & \mathrm{Ai}'(\omega_3 s) \end{pmatrix} \begin{pmatrix} e^{-i\pi/6} & 0 \\ 0 & e^{i\pi/6} \end{pmatrix}, \qquad s \in IV,
\end{aligned}$$

where $\omega_3 = e^{2\pi i/3}$ is a primitive third root of unity.

With the above definitions of $\Psi^{(1)}$ and f_1 it may then be shown that for any analytic prefactor $E^{(1)}$ the matrix $M^{(1)}$ defined by

$$\begin{aligned}
M^{(1)}(z) &= E^{(1)}(z)\Psi^{(1)}((n+1)^{2/3} f_1(z)) \\
&\quad \times \begin{pmatrix} z^{-1/2}e^{-z}s_n^{1/2}(z)e^{(n+1)\varphi_P(z)} & 0 \\ 0 & z^{1/2}s_n^{-1/2}(z)e^{z}e^{-(n+1)\varphi_P(z)} \end{pmatrix}. \qquad (6.23)
\end{aligned}$$

satisfies the jump conditions on Γ_s, where s is any of the symbols P, P^-, P^+, and R. The extra factor $E^{(1)}$ has to be chosen in such a way that $M^{(1)}$ satisfies the matching condition on Γ_1 as well. We choose

$$\begin{aligned}
E^{(1)}(z) &= \sqrt{\pi}e^{i\pi/6}N(z) \begin{pmatrix} z^{1/2}e^{z}s_n^{-1/2}(z) & 0 \\ 0 & z^{-1/2}e^{-z}s_n^{1/2}(z) \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} \begin{pmatrix} ((n+1)^{2/3} f_1(z))^{\frac{1}{4}} & 0 \\ 0 & ((n+1)^{2/3} f_1(z))^{-\frac{1}{4}} \end{pmatrix}. \qquad (6.24)
\end{aligned}$$

On the part of Γ_1 that lies in $(f_1)^{-1}(I)$ (the arc between Γ_{P-} and Γ_R) the asymptotic expansions for the Airy functions (2.32) and (2.33) give

$$\begin{aligned}
&\mathrm{Ai}((n+1)^{2/3} f_1(z))\\
&\quad= \frac{1}{2\sqrt{\pi}}(n+1)^{-1/6}(f_1(z))^{-\frac{1}{4}} e^{-(n+1)\varphi_P(z)}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),\\
&\mathrm{Ai}((n+1)^{2/3}\omega_3^2 f_1(z))\\
&\quad= \frac{1}{2\sqrt{\pi}}(n+1)^{-1/6}(f_1(z))^{-\frac{1}{4}} e^{i\pi/6}e^{(n+1)\varphi_P(z)}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),\\
&\mathrm{Ai}'((n+1)^{2/3} f_1(z))\\
&\quad= \frac{-1}{2\sqrt{\pi}}(n+1)^{1/6}(f_1(z))^{\frac{1}{4}} e^{-(n+1)\varphi_P(z)}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),\\
&\mathrm{Ai}'((n+1)^{2/3}\omega_3^2 f_1(z))\\
&\quad= \frac{-1}{2\sqrt{\pi}}(n+1)^{1/6}(f_1(z))^{\frac{1}{4}} e^{-i\pi/6}e^{(n+1)\varphi_P(z)}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right).
\end{aligned}$$

Here the fourth root in $(f_1(z))^{\frac{1}{4}}$ is defined with a cut along Γ_P. On this part of Γ_1, we have

$$\begin{aligned}
M^{(1)}(z) &= \frac{1}{2\sqrt{\pi}} E^{(1)}(z)\begin{pmatrix}(n+1)^{-1/6} & 0\\ 0 & (n+1)^{1/6}\end{pmatrix}\\
&\times\begin{pmatrix}(f_1(z))^{-\frac{1}{4}} e^{-(n+1)\varphi_P(z)}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) & -(f_1(z))^{-\frac{1}{4}} e^{i\pi/6}e^{(n+1)\varphi_P(z)}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\\ -(f_1(z))^{\frac{1}{4}} e^{-(n+1)\varphi_P(z)}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) & -(f_1(z))^{\frac{1}{4}} e^{i\pi/6}e^{(n+1)\varphi_P(z)}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\end{pmatrix}\\
&\times\begin{pmatrix}e^{-i\pi/6}z^{-1/2}e^{-z}s_n^{1/2}(z)e^{(n+1)\varphi_P(z)} & 0\\ 0 & e^{i\pi/6}z^{1/2}s_n^{-1/2}(z)e^{z}e^{-(n+1)\varphi_P(z)}\end{pmatrix}.
\end{aligned}$$

After plugging the expression (6.24) of $E^{(1)}(z)$ and performing straightforward computations, we get

$$M^{(1)}(z) = N(z)\begin{pmatrix}1+\mathcal{O}\left(\frac{1}{n}\right) & ze^{2z}s_n^{-1}(z)\mathcal{O}\left(\frac{1}{n}\right)\\ z^{-1}e^{-2z}s_n(z)\mathcal{O}\left(\frac{1}{n}\right) & 1+\mathcal{O}\left(\frac{1}{n}\right).\end{pmatrix}$$

We now check that the matching condition (6.22) indeed holds true. In view of the previous expression for $M^{(1)}(z)$, it is equivalent to show that the quotients $s_n^{-1}(z)N_{21}(z)/N_{12}(z)$, $s_n^{-1}(z)N_{11}(z)/N_{22}(z)$, and their inverses, are all of order $\mathcal{O}(n^{(1-\alpha)/2})$ uniformly for $z\in\Gamma_1$ as n becomes large. Let us look at the first expression $s_n^{-1}(z)N_{21}(z)/N_{12}(z)$. Making use of relations (6.2)–(6.5), we are lead to study the order of the quotient

$$\begin{aligned}
\frac{\widehat{s}_n(-2)\widetilde{s}_n(-2)\widehat{s}_n(1/\psi_P(z))\widetilde{s}_n(-4\psi_Q(z))}{s_n(z)} &= s_n^2(-2)s_n\left(\frac{-z}{\sqrt{1+z^2}}\right)\\
&\quad\times\left(1+\mathcal{O}\left(\frac{\log^2 n}{n}\right)\right),
\end{aligned}\tag{6.25}$$

where, in the last equality, we have successively used the relations (2.38), $-4\psi_P(z)\psi_Q(z)=1$, (6.19), (6.18), (6.17), and the explicit expressions (2.11) for ψ_P and ψ_Q. We have $|z-z_1|=\delta_n$ which, by hypothesis, is of order ρ_n^{-2}. Then, $|-z/\sqrt{1+z^2}|\geqslant\frac{1}{2\delta_n^{1/2}}$ for n large.

Applying the first estimate in (6.20), we deduce that, as $n \to \infty$,

$$s_n\left(\frac{-z}{\sqrt{1+z^2}}\right) = \mathcal{O}\left(n^{(1-\alpha)\delta_n^{1/2}}\right) = \mathcal{O}(1), \tag{6.26}$$

uniformly for $|z - z_1| = \delta_n$. Since $s_n^2(-2) = \mathcal{O}(n^{(1-\alpha)/2})$, we get in view of (6.25) and (6.26) that $s_n^{-1}(z)N_{21}(z)/N_{12}(z)$ is of order $\mathcal{O}(n^{(1-\alpha)/2})$. The three others expressions may be handled in the same way. The matching condition (6.22) is proved.

From (6.24) it is easy to see that $E^{(1)}$ is analytic in $\Delta_1 \setminus \Gamma_P$. On Γ_P, both N and $(f_1)^{\frac{1}{4}}$ have a jump. N has the jump (6.1) and $(f_1)^{\frac{1}{4}}$ satisfies $(f_1)_+^{\frac{1}{4}} = -i\,(f_1)_-^{\frac{1}{4}}$. Straightforward calculations then show that $E_+^{(1)} = E_-^{(1)}$ on Γ_P, so that $E^{(1)}$ is analytic across Γ_P. From (6.24) and the fact that the entries of N have at most fourth root singularities at z_1, see (6.13), we see that the entries of $E^{(1)}$ have at most a square root singularity at z_1. Since $E^{(1)}$ is analytic in $\Delta_1 \setminus \{z_1\}$, the singularity at z_1 is removable, and this proves that $E^{(1)}$ is analytic in the full Δ_1. This completes the description of the parametrix $M^{(1)}$ in the neighborhood Δ_1 of z_1.The expression (6.23) of the parametrice $M^{(1)}$ will be used when computing the asymptotics stated in Theorem 2.13 and Corollary 2.14.

In a similar way, we can construct a parametrice $M^{(2)}$ near the other branch point z_2.

6.3. *Third transformation*

We now introduce the final matrix

$$S(z) = \begin{cases} T(z)\,(N(z))^{-1}, & z \text{ outside } \Gamma_1 \text{ and } \Gamma_2, \\ T(z)\left(M^{(j)}(z)\right)^{-1}, & z \text{ inside } \Gamma_1 \text{ or } \Gamma_2, \end{cases} \tag{6.27}$$

where the contours Γ_1 and Γ_2 (which depend on n if $\alpha < 1$) were defined in the previous section. Inside Γ_1 or Γ_2 the matrices T and $M^{(j)}$ have the same jumps, hence S has no jumps inside Γ_j, $j = 1, 2$. Outside Γ_1 and Γ_2 the matrices T and N have the same jump matrices on Γ_P. Hence S has no jump on Γ_P. This means that S solves a Riemann–Hilbert problem on the system of curves shown in Fig. 7.

S is analytic outside the above system of contours and it is normalized at infinity

$$S(z) = I + \mathcal{O}\left(\frac{1}{z}\right), \qquad z \to \infty. \tag{6.28}$$

Theorem 6.4. *The matrix $S(z)$ has the behavior*

$$S(z) = I + \mathcal{O}\left(\frac{1}{n^{(\alpha+1)/2}}\right), \qquad n \to \infty, \tag{6.29}$$

uniformly on $\mathbb{C} \setminus \Sigma_S$, where Σ_S are the contours in Fig. 7.

Proof. The jumps on all of the contours are uniformly of the form $I + \mathcal{O}(e^{-cn})$ with some fixed $c > 0$, except for the jumps on the circles Γ_j where we have

$$S_+(z) = S_-(z)M^{(j)}(z)N^{-1}(z), \qquad z \in \Gamma_j.$$

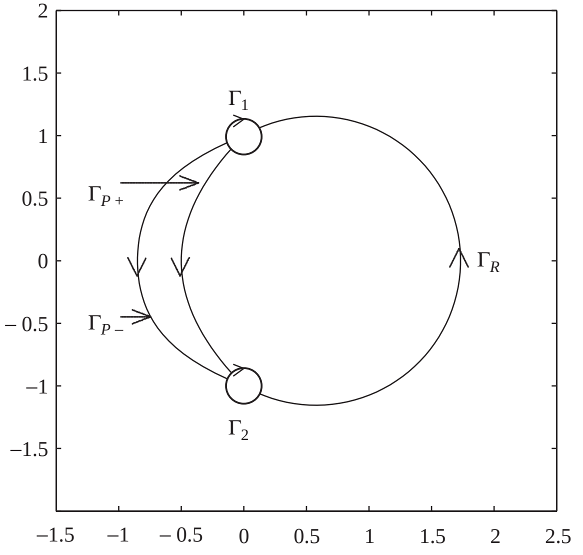

Fig. 7. Contours of the RHP for S.

Because of the matching condition we have

$$M^{(j)}(z)N^{-1}(z) = I + \mathcal{O}\left(\frac{1}{n^{(\alpha+1)/2}}\right)$$

uniformly for $z \in \Gamma_j$. Hence $S(z)$ solves a Riemann–Hilbert problem, normalized at ∞ with jumps close to the identity matrix up to $\mathcal{O}(1/n^{(\alpha+1)/2})$, uniformly on the contours Σ_S. We can then use arguments as those leading to Theorem 7.171 in [10] to obtain (6.29). We can also use Theorem 3.1 of [22], whose elementary complex analysis proof is due to Aptekarev [1]. Note that, in our situation, the contour Σ_S is not simple and also varies (slowly) with n. One checks that the proof of [22, Theorem 3.1] can be adapted to the present case.

7. Proofs of the asymptotic formulas

We now know the asymptotic behavior (6.29) of S as $n \in \Lambda$ and $n \to \infty$. We will trace back our steps to the original Riemann–Hilbert problem for Y to obtain asymptotics for the scaled polynomials in the rational interpolation to the exponential function.

7.1. Proofs of Theorems 2.10, 2.1, and Corollary 2.11

Proof of Theorem 2.10 when $n \in \Lambda$. We recall that Λ denotes the set of normal indices with respect to the scheme B, namely the indices n such that $\deg Q_n = n$, see (5.1). We start with the proof of the asymptotic formula (2.23) for P_n. Let K be a compact subset of $\mathbb{C} \setminus \Gamma_P$. We have the freedom to take the contours Γ_{P^-}, Γ_{P^+} near Γ_P, and the circles Γ_1 and Γ_2 around z_1 and z_2 in such a way that K is in the exterior of these curves. Let $z \in K$. Then we follow the transformations $Y \mapsto U \mapsto T \mapsto S$. We see first from (5.4) that

$$P_n(z) = Y_{21}(z) = U_{21}(z)e^{-2(n+1)\ell}e^{(n+1)g_P(z)}.$$

Then from the definition of T in (5.12) and (5.13), we get that $U_{21}(z) = T_{21}(z)$. We finally note that $T(z) = S(z)N(z)$, since z is outside Γ_1 and Γ_2. For $T_{21}(z)$, we get

$$T_{21}(z) = S_{21}(z)N_{11}(z) + S_{22}(z)N_{21}(z).$$

Since $S = I + \mathcal{O}\left(1/n^{(\alpha+1)/2}\right)$, we get

$$P_n(z) = e^{-2(n+1)\ell}e^{(n+1)g_P(z)}\left(N_{21}(z) + N_{21}(z)\mathcal{O}\left(\frac{1}{n^{(\alpha+1)/2}}\right)\right.$$
$$\left. + N_{11}(z)\mathcal{O}\left(\frac{1}{n^{(\alpha+1)/2}}\right)\right)$$

uniformly on K. From the expressions (6.2)–(6.4) and relations (2.38), we see that, locally uniformly, $N_{11}(z)$ is of the same order as $s_n^{-2}(-2)N_{21}(z)$ which, in view of (6.18) and (6.20) applied with $z = 2$, leads to

$$N_{11}(z) = N_{21}(z)\mathcal{O}\left(n^{(1-\alpha)/2}\right). \tag{7.1}$$

Hence

$$P_n(z) = N_{21}(z)e^{-2(n+1)\ell}e^{(n+1)g_P(z)}\left(1 + \mathcal{O}\left(\frac{1}{n^{\alpha}}\right)\right)$$

uniformly on K. Now we use the formula in (6.16) for N_{21}, the relations (2.38), and we recall that $\ell = -\pi i/2$, to obtain (2.23).

For Q_n we proceed differently, because of the way Q_n appears in the entries of Y. We take $z_0 \in \mathbb{C} \setminus \Gamma_Q$, and show that there is a neighborhood Δ of z_0 such that (2.24) holds uniformly for $z \in \Delta$. First we assume that z_0 belongs to the outside region $\mathbb{C} \setminus \overline{D_0}$. Then we can take the original contour Γ so that a neighborhood Δ of z_0 is in the outside region. Then for $z \in \Delta$, we have by (5.2)

$$Q_n(z) = \Omega_n(z)Y_{22}(z) = U_{22}(z)\Omega_n(z)e^{-(n+1)g_P(z)}.$$

We can open the contour around Γ_P so that Δ is in the exterior region to this contour. Then we have $U = T = SN = \left(I + \mathcal{O}\left(1/n^{(\alpha+1)/2}\right)\right)N$, so that $U_{22}(z) = N_{22}(z) + N_{22}(z)\mathcal{O}\left(1/n^{(\alpha+1)/2}\right) + N_{12}(z)\mathcal{O}\left(1/n^{(\alpha+1)/2}\right)$. In the same manner we proved (7.1), we can show that, locally uniformly,

$$N_{12}(z) = N_{22}(z)\mathcal{O}\,(n^{(1-\alpha)/2}). \tag{7.2}$$

Hence, $U_{22}(z) = N_{22}(z)\left(1 + \mathcal{O}\left(1/n^{\alpha}\right)\right)$ uniformly for $z \in \Delta$. This leads to the second formula in (2.24), if we use the formula for N_{22} in (6.16), the relations (2.38), the fact that $-4\psi_P(z)\psi_Q(z) = 1$ and relation (6.19).

If $z_0 \in D_0$, then we can also open up the lenses around Γ_P so that a neighborhood Δ of z_0 is not contained in these lenses. Then we have for $z \in \Delta$, by (5.3)

$$Q_n(z) = e^{-nz}E_n(z) - P_n(z)e^{-2nz}$$

and we need to find out, what is the dominant contribution as n gets large. We already have the asymptotic formula for P_n, from which it follows that

$$\frac{1}{n}\,\log|P_n(z)e^{-2nz}| \to \mathrm{Re}\,(g_P(z)-2z), \tag{7.3}$$

For E_n it is easy to obtain in a similar way

$$\frac{1}{n}\,\log|E_n(z)e^{-nz}| \to \mathrm{Re}\,(2\,\log\,z-g_P(z)). \tag{7.4}$$

Now it turns out that for $z\in D_0$ the term $P_n(z)e^{-2nz}$ dominates. Indeed, we have by (4.6)

$$\mathrm{Re}\,(2g_P(z)-2z-2\,\log\,z)=-2\,\mathrm{Re}\,(\varphi_P(z))$$

and we know that the real part of φ_P is negative in D_0. In view of (7.3) and (7.4) we then obtain the formula for Q_n

$$Q_n(z)=-P_n(z)e^{-2nz}\left(1+\mathcal{O}\,\left(\frac{1}{n}\right)\right)$$

uniformly for $z\in\Delta$, and from what we already know about P_n, we get the first line in formula (2.24).

We finally have to consider the case that z_0 is on Γ_P (but not one of the branch points). After opening up lenses around Γ_P we have the contour Γ_{P-} to the left of z_0, and the contour Γ_{P+} to the right. We can take a neighborhood Δ of z_0 that is strictly contained in the domain bounded by Γ_{P-} and Γ_{P+}. Then for $z\in\Delta\cap D_{\infty,P}$, we have

$$Q_n(z)=\Omega_n(z)Y_{22}(z)=U_{22}(z)\Omega_n(z)e^{-(n+1)g_P(z)}$$

and $U(z)=T(z)V_{P-}^{-1}(z)$. Now we have $U_{22}(z)=T_{22}(z)$, since the second column of V_{P-} is simply $\begin{pmatrix}0 & 1\end{pmatrix}^T$. We open the circles Γ_1 and Γ_2 around z_1 and z_2 so that Δ is in the exterior. Then $T=SN=(I+\mathcal{O}\,(1/n^{(\alpha+1)/2}))N$, and so

$$\begin{aligned}Q_n(z)=\Omega_n(z)e^{-(n+1)g_P(z)}&\Big(N_{22}(z)+N_{22}(z)\mathcal{O}\,\left(1/n^{(\alpha+1)/2}\right)\\&+N_{12}(z)\mathcal{O}\,\left(1/n^{(\alpha+1)/2}\right)\Big)\end{aligned}$$

uniformly for $z\in\Delta\cap D_{\infty,P}$. Using the estimate (7.2), the formula (6.16) for N_{22}, and the relations (2.38), we then find the second formula in (2.24).

For $z\in\Delta\cap D_0$, we have

$$\begin{aligned}Q_n(z)&=e^{-nz}E_n(z)-P_n(z)e^{-2nz}\\&=\Omega_n(z)Y_{22}(z)-e^{-2nz}Y_{21}(z)\\&=\Omega_n(z)e^{-(n+1)g_P(z)}U_{22}(z)-e^{-2nz}e^{(n+1)(g_P(z)-2\ell)}U_{21}(z).\end{aligned}$$

Now we have that

$$U(z)=T(z)V_{P+}(z),$$

with V_{P+} given by (5.14). Then

$$\begin{aligned}Q_n(z)&=\Omega_n(z)e^{-(n+1)g_P(z)}T_{22}(z)\\&\quad-e^{-2nz}e^{(n+1)(g_P(z)-2\ell)}\left[T_{21}(z)+T_{22}(z)(z^{-1}e^{-2z}e^{2(n+1)\varphi_P(z)}s_n(z))\right].\end{aligned}$$

The factor multiplying $T_{22}(z)$ is exactly zero. This follows from (4.6). Thus only the second term remains. This gives

$$\begin{aligned} Q_n(z) &= -e^{(n+1)(g_P(z)-2\ell)} e^{-2nz} T_{21}(z) \\ &= -e^{(n+1)(g_P(z)-2\ell)} e^{-2nz} \left(N_{21}(z) + N_{21}(z)\mathcal{O}\left(1/n^{(\alpha+1)/2}\right) \right. \\ &\quad \left. + \ N_{11}(z)\mathcal{O}\left(1/n^{(\alpha+1)/2}\right)\right), \end{aligned}$$

uniformly for $z \in \Delta \cap D_0$. Using the uniform estimate (7.1), the expression for N_{21}, and the relations (2.38), we get the asymptotic formula for $z \in \Delta \cap D_0$. Note that, by (4.7), the limit of the first formula in (2.24) on the +side of Γ_P agree with the limit of the second formula on the −side of Γ_P. Hence, both asymptotic formulas in (2.24) extend to $\Gamma_P \setminus \{z_1, z_2\}$.

For E_n we take $z_0 \in \mathbb{C} \setminus \Gamma_E$, and show that there is a neighborhood Δ of z_0 such that (2.25) holds uniformly for $z \in \Delta$. First we assume that z_0 belongs to the inside region D_0. Then for $z \in \Delta$, we have by (5.3) and (5.4)

$$E_n(z) = \Omega_n(z)e^{nz}Y_{22}(z) = U_{22}(z)\Omega_n(z)e^{nz}e^{-(n+1)g_P(z)}.$$

We can open the contour around Γ_P so that Δ is in the exterior region to this contour. Then we have $U = T = SN = \left(I + \mathcal{O}\left(1/n^{(\alpha+1)/2}\right)\right)N$, so that $U_{22}(z) = N_{22}(z) + N_{22}(z)\mathcal{O}\left(1/n^{(\alpha+1)/2}\right) + N_{12}(z)\mathcal{O}\left(1/n^{(\alpha+1)/2}\right)$ uniformly for $z \in \Delta$. Using the estimate (7.2), the formula for $N_{22}(z)$ in (6.16), the relations (2.38) and (6.19), we obtain the second formula in (2.25) for $z \in D_0$.

If $z_0 \in D_{\infty,P}$, then we can also open up the lens around Γ_P so that a neighborhood Δ of z_0 is not contained in this lens. Then we have for $z \in \Delta$, by (1.3)

$$E_n(z) = P_n(z)e^{-nz} + Q_n(z)e^{nz} \tag{7.5}$$

and we need to find out, what is the dominant contribution as n gets large. We already have asymptotic formulas for P_n and Q_n, from which it follows that

$$\frac{1}{n}\log|P_n(z)e^{-nz}| \to \mathrm{Re}\,(g_P(z) - z), \tag{7.6}$$

$$\frac{1}{n}\log|Q_n(z)e^{nz}| \to \mathrm{Re}\,(-g_P(z) + 2\log(z) + z). \tag{7.7}$$

Now it turns out that for $z \in D_{\infty,P}$ the term $P_n(z)e^{-nz}$ dominates. Indeed, as before, we have by (4.6)

$$\mathrm{Re}\,(2g_P(z) - 2z - 2\log(z)) = -2\,\mathrm{Re}\,(\varphi_P(z))$$

and we know, by Lemma 2.9 that the real part of φ_P is negative in $D_{\infty,P}$. In view of (7.6) and (7.7) we then obtain the formula for E_n

$$E_n(z) = P_n(z)e^{-nz}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)$$

uniformly for $z \in \Delta$, and from what we already know about P_n,

$$E_n(z) = -\frac{(-1)^{n+1}\sqrt{2}s_n(1/\psi_P(z))e^{n(g_P(z)-z)}}{s_n(2)\sqrt{4\psi_P^2+1}}\left(1 + \mathcal{O}\left(\frac{1}{n^{\alpha}}\right)\right)$$

uniformly for $z \in \Delta$. This is the first line in formula (2.25). The proof for the case $z_0 \in D_{\infty,Q}$ is similar. It uses the fact that the dominant term in (7.5) for $z \in D_{\infty,Q}$ is $Q_n(z)e^{nz}$. Actually, it is simply an analytic continuation of the formula already obtained in the domain D_0. This completes the proof of Theorem 2.10 when $n \in \Lambda$.

Proof of Theorem 2.1. Since different schemes of points enter the proof, we will keep track of these schemes by using a superscript. Hence, we introduce the notation P_n^B, Q_n^B, E_n^B and Ω_n^B to specify that the interpolants are taken with respect to the scheme B. The scheme $-B$ consists of the negatives of the points of B. We first need a preliminary result. We claim that for all normal indices $n \in \Lambda$, large enough, we have $\deg P_n^B = n$. Indeed, the formula (2.23) of Theorem 2.10 allows one to apply Rouché's theorem on a circle $C(0, R)$ containing the curve Γ_P, showing that, for n large, the difference between the number of poles and zeros of P_n^B outside $C(0, R)$ is equal to the same difference for the function given by the ratio in the right-hand side of (2.23). This ratio has no zero outside $C(0, R)$ but has a pole of order n at infinity, since $g_P(z) = \log z + \mathcal{O}(1/z)$ as $z \to \infty$. Hence, since $\deg P_n^B \leqslant n$, the same conclusion holds for P_n^B, namely P_n^B has no zero outside $C(0, R)$ but has a pole of order n at infinity. In particular, for n large, P_n^B is of exact degree n. This proves the claim.

Now, we proceed by contradiction, assuming that the assertion of Theorem 2.1 is false, namely there exists an infinite sequence $\mathcal{V}$ of indices n with

$$\deg p_n^B < n \text{ or } \deg q_n^B < n,$$

or equivalently

$$\deg P_n^B < n \text{ or } \deg Q_n^B < n.$$

Assume that the pair $(P_n^B(z), Q_n^B(z))$ is minimal among all solutions of the interpolation problem of type (n, n), in the sense that the fraction P_n^B/Q_n^B is irreducible. We set

$$\deg P_n^B = n - \nu, \quad \deg Q_n^B = n - \mu,$$

with $\nu > 0$ or $\mu > 0$. First, assume $\mu \leqslant \nu$. By the previous claim, for n large, the case $\mu = 0$ is impossible. Moreover, the pair $(P_n^B(z), Q_n^B(z))$ solves the rational interpolation problem of type $(n-\mu, n-\mu)$ associated to any subset $C^{(2n-2\mu)}$ of $2n-2\mu+1$ points of $B^{(2n)}$. Second, assume $\mu > \nu$. Then, considering the scheme $-B$ instead of the scheme B, and changing z into $-z$ in relation (1.3), the previous claim again shows, that for n large, the case $\nu = 0$ is impossible. Moreover, the pair $(Q_n^B(-z), P_n^B(-z))$ solves the rational interpolation problem of type $(n-\nu, n-\nu)$ associated to any subset $C^{(2n-2\nu)}$ of $2n-2\nu+1$ points of $-B^{(2n)}$. In both cases, the degrees $n-\mu$ or $n-\nu$ are normal with respect to the sets of points $C^{(2n-2\mu)}$ or $C^{(2n-2\nu)}$. Note also that the corresponding error function E_n^B has $2n+1 > 2n-2\mu+1$ (resp. $2n+1 > 2n-2\nu+1$) zeros in some given disk $D(0, \varepsilon)$, $\varepsilon > 0$, around the origin. For each index $n \in \mathcal{V}$, the set $C^{(2n-2\mu)}$ or $C^{(2n-2\nu)}$, according to $\mu \leqslant \nu$ or $\mu > \nu$, is a row of a new scheme $C := C^{(2l)}$, $l = l_n$, $n \in \mathcal{V}$. Note that $l_n \to \infty$ as $n \to \infty$. Indeed, it is well known that an exponential polynomial $Q(z)e^z + P(z)$ has no more than $2\deg P + 2\deg Q + c$ zeros in a given compact set $\mathcal{K}$ of $\mathbb{C}$ where c is a constant which depends only on $\mathcal{K}$, see e.g. [43]. By construction, each index l_n is normal with respect to the scheme C. Hence, the asymptotic formulas in Theorem 2.10 applies. In particular, (2.25) implies that, for l large enough, $E_l^C(z) = E_n^B(z)$ has exactly $2l+1$ zeros in the disk

$D(0, \varepsilon)$, namely the zeros of $\Omega_l^C(z)$. This contradicts the fact that E_n^B has $2n + 1 > 2l + 1$ zeros in $D(0, \varepsilon)$, and the proof of Theorem 2.1 is completed.

Completion of the proof of Theorem 2.10. We just proved that for any triangular scheme $B := B^{(2n)}$, $n = n_\nu$, of points lying in a given compact set of $\mathbb{C}$, all indices n, large enough, are normal with respect to rational interpolation to the exponential function. Hence, Theorem 2.10 is actually valid for any sequence of indices $n \in \mathbb{N}$.

Proof of Corollary 2.11. It is straightforward from the relation (1.3) and the fact that the polynomial Q_n^B is monic that the following symmetry relation holds:

$$Q_n^{-B}(z) = \frac{P_n^B(-z)}{(-1)^n \alpha_n^B}.$$

Hence $\alpha_n^B = (-1)^n P_n^B(0)/Q_n^{-B}(0)$, and (2.27) follows from evaluating (2.23) and the first equality in (2.24) at $z = 0$. We also use with $z = 2$ the fact that for n large,

$$s_n^{-B}(z) = \frac{1}{s_n^B(z)}\left(1 + \mathcal{O}\left(\frac{\rho_n^2}{n}\right)\right), \qquad z \in \overline{\mathbb{C}} \setminus \{0\}.$$

Hence, in view of (2.1), we have that

$$s_n^{-B}(2) = \frac{1}{s_n^B(2)}\left(1 + \mathcal{O}\left(\frac{\log^2 n}{n}\right)\right), \qquad \text{if } 0 < \alpha < 1$$

and

$$s_n^{-B}(2) = \frac{1}{s_n^B(2)}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \qquad \text{if } \alpha = 1.$$

7.2. Proofs of Theorems 2.12, 2.13, and Corollary 2.14

Proof of Theorem 2.12. We will not give details of the proof since the asymptotic formulas for Q_n and E_n can be obtained as in Theorem 2.9 of [27]. The asymptotic formula for P_n is derived as in Theorem 2.10 of [27] where we now use the fact that $N_{11}/N_{21}(z)$ and $N_{12}/N_{22}(z)$ are locally uniformly of order $\mathcal{O}(n^{(1-\alpha)/2})$ as $n \to \infty$, see (7.1) and (7.2).

Proof of Theorem 2.13. If we unravel all the transformations for $z \in \Delta_1$, we find that $Y(z)$ is a product of matrices, the exact number of which depends on the region where z is, namely $f^{-1}(I)$, $f^{-1}(II)$, $f^{-1}(III)$, or $f^{-1}(IV)$, see Fig. 6. Considering the (2, 1) entry of $Y(z)$, we get after some calculations independent from the considered region, that for $z \in \Delta_1$,

$$\begin{aligned}
&(-1)^{n+1} P_n(z) \\
&\quad = \sqrt{\pi} e^{(n+1)(g_P(z)+\varphi_P(z))} \\
&\qquad \times \Bigg[\sum_{k=1}^{2} S_{2k}(z) \left(N_{k1}(z) + i z^{-1} e^{-2z} s_n(z) N_{k2}(z) \right) (n+1)^{1/6} f_1(z)^{1/4} \\
&\qquad \times \mathrm{Ai}((n+1)^{2/3} f_1(z)) + \sum_{k=1}^{2} S_{2k}(z) \left(-N_{k1}(z) + i z^{-1} e^{-2z} s_n(z) N_{k2}(z) \right) \\
&\qquad \times (n+1)^{-1/6} f_1(z)^{-1/4} \mathrm{Ai}'((n+1)^{2/3} f_1(z)) \Bigg].
\end{aligned} \tag{7.8}$$

From the jump condition (6.1) for N on Γ_P it easily follows that for each k,

$$\begin{aligned}
&\left(N_{k1}(z) + i z^{-1} e^{-2z} s_n(z) N_{k2}(z) \right)_+ \\
&\quad = i \left(N_{k1}(z) + i z^{-1} e^{-2z} s_n(z) N_{k2}(z) \right)_-, \qquad z \in \Gamma_P.
\end{aligned}$$

The fourth root in $f_1(z)^{1/4}$ is defined with a cut along Γ_P, and on Γ_P there is a jump

$$\left(f_1(z)^{1/4} \right)_+ = -i \left(f_1(z)^{1/4} \right)_-, \qquad z \in \Gamma_P.$$

Thus the products

$$\left(N_{k1}(z) + i z^{-1} e^{-2z} s_n(z) N_{k2}(z) \right) f_1(z)^{1/4}$$

are analytic across Γ_P. Similarly, we have that the products

$$\left(-N_{k1}(z) + i z^{-1} e^{-2z} s_n(z) N_{k2}(z) \right) f_1(z)^{-1/4}$$

are analytic across Γ_P.

Now we recall that $S(z) = I + \mathcal{O}\left(1/n^{(\alpha+1)/2}\right)$. Consequently, (7.8) can be rewritten in Δ_1 as

$$\begin{aligned}
(-1)^{n+1} P_n(z) = \sqrt{\pi} e^{(n+1)(g_P(z)+\varphi_P(z))} \Big[& n^{1/6} f_1(z)^{1/4} \mathrm{Ai}((n+1)^{2/3} f_1(z)) \\
& \times \Big((N_{21}(z) + i z^{-1} e^{-2z} s_n(z) N_{22}(z)) \left(1 + \mathcal{O}\left(1/n^{(\alpha+1)/2}\right) \right) \\
& + \left(N_{11}(z) + i z^{-1} e^{-2z} s_n(z) N_{12}(z) \right) \mathcal{O}\left(1/n^{(\alpha+1)/2}\right) \Big) \\
& + n^{-1/6} f_1(z)^{-1/4} \mathrm{Ai}'((n+1)^{2/3} f_1(z)) \\
& \times \Big((-N_{21}(z) + i z^{-1} e^{-2z} s_n(z) N_{22}(z)) \left(1 + \mathcal{O}\left(1/n^{(\alpha+1)/2}\right) \right) \\
& + \left(-N_{11}(z) + i z^{-1} e^{-2z} s_n(z) N_{12}(z) \right) \mathcal{O}\left(1/n^{(\alpha+1)/2}\right) \Big) \Big].
\end{aligned} \tag{7.9}$$

By use of (7.1) and (7.2), the previous expression reduces to

$$\begin{aligned}(-1)^{n+1}P_n(z) = \sqrt{\pi}e^{(n+1)(g_P(z)+\varphi_P(z))}\Big[& n^{1/6}f_1(z)^{1/4}\mathrm{Ai}((n+1)^{2/3}f_1(z)) \\ & \times(N_{21}(z)+iz^{-1}e^{-2z}s_n(z)N_{22}(z))\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right) \\ & + n^{-1/6}f_1(z)^{-1/4}\mathrm{Ai}'((n+1)^{2/3}f_1(z))(-N_{21}(z) \\ & + iz^{-1}e^{-2z}s_n(z)N_{22}(z))\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right)\Big].\end{aligned}$$

This proves the asymptotic formula for P_n near z_1. The asymptotic formulas for Q_n and E_n are obtained in the same way.

Before proving Corollary 2.14, the following lemma is needed.

Lemma 7.1. *Let $\delta_n > 0$ be a sequence of real numbers such that $\delta_n = o(1/\log^2 n)$ as $n \to \infty$. The function h_1 (which depends on n) has no zeros in $|z - z_1| < \delta_n$ as $n \to \infty$. Moreover, the functions h_1 and h_2 satisfy, as $n \to \infty$,*

$$h_2(z)/h_1(z) = \mathcal{O}(\log n), \quad \textit{if} \quad \alpha < 1, \tag{7.10}$$

and

$$h_2(z)/h_1(z) = \mathcal{O}(1), \quad \textit{if} \quad \alpha = 1, \tag{7.11}$$

uniformly in $|z - z_1| < \delta_n$. Similar property and estimates hold near the other branch point z_2.

Proof. In view of the definitions of the functions h_1 and h_2 in Theorem 2.13, along with the expressions (6.16) of N_{21} and N_{22}, the quotient $h_2(z)/h_1(z)$ is equal to the quotient of the two following expressions

$$\left[-\frac{\widehat{s}_n(1/\psi_P(z))e^{-g_P(z)}}{\sqrt{4\psi_P^2(z)+1}} + i\,\frac{s_n(z)e^{g_P(z)-2z-2\log z}}{\widetilde{s}_n(-4\psi_Q(z))\sqrt{4\psi_Q^2(z)+1}}\right] f_1(z)^{1/4}$$

and

$$\left[\frac{\widehat{s}_n(1/\psi_P(z))e^{-g_P(z)}}{\sqrt{4\psi_P^2(z)+1}} + i\,\frac{s_n(z)e^{g_P(z)-2z-2\log z}}{\widetilde{s}_n(-4\psi_Q(z))\sqrt{4\psi_Q^2(z)+1}}\right] f_1(z)^{-1/4}.$$

Now, since f_1 is analytic with a simple zero at z_1, $f_1(z)/(z - z_1)$ is locally bounded near z_1. Factorizing $e^{-g_P(z)}$, making use of relations (4.5) and (4.6), and finally multiplying both expressions with

$$\widetilde{s}_n(-4\psi_Q(z))\sqrt{4\psi_P^2(z)+1}\Big/ s_n(z)(z-z_1)^{1/4}, \tag{7.12}$$

we may equivalently compare

$$-\frac{\widehat{s}_n(1/\psi_P(z))\widetilde{s}_n(-4\psi_Q(z))}{s_n(z)} - ie^{-2\varphi_P(z)}\sqrt{\frac{4\psi_P^2(z)+1}{4\psi_Q^2(z)+1}} \tag{7.13}$$

and

$$\left(\frac{\widehat{s}_n(1/\psi_P(z))\widetilde{s}_n(-4\psi_Q(z))}{s_n(z)} - ie^{-2\varphi_P(z)}\sqrt{\frac{4\psi_P^2(z)+1}{4\psi_Q^2(z)+1}}\right)(z-z_1)^{-1/2}. \tag{7.14}$$

First, for $|z-z_1| < \delta_n$, we show that the modulus of the expression (7.13) admit an uniform lower bound independent of n. Since (7.12) has no poles in a given neighborhood of z_1, it will in particular imply that h_1 has no zeros in $|z-z_1| < \delta_n$.

From (6.25), where we have taken out the factors $\widehat{s}_n(-2)\widetilde{s}_n(-2)$ on the left and $s_n^2(-2)$ on the right, we know that the first term in (7.13) can be written as

$$-\frac{\widehat{s}_n(1/\psi_P(z))\widetilde{s}_n(-4\psi_Q(z))}{s_n(z)} = -s_n\left(\frac{-z}{\sqrt{1+z^2}}\right)\left(1+\mathcal{O}\left(\frac{\log^2 n}{n}\right)\right). \tag{7.15}$$

Let $z-z_1 = h$, $|h| < \delta_n$. Then, $|-z/\sqrt{1+z^2}| \geqslant \frac{1}{2|h|^{1/2}}$ for n large. Applying inequality (6.21), we deduce that, as $n \to \infty$,

$$s_n\left(\frac{-z}{\sqrt{1+z^2}}\right) = 1 + \mathcal{O}\left(|h|^{1/2}\rho_n\right) = 1 + o(1), \tag{7.16}$$

uniformly in $|z-z_1| < \delta_n$, where in the last equality, we have used (2.1) and the assumption that $\delta_n = o(1/\log^2 n)$. Since the second term in (7.13) tends to -1 as z tends to z_1, the expression in (7.13) tends to -2 and is thus bounded below, for n large.

Second, we show that the modulus of the expression (7.14) is dominated by $\log n$ (if $\alpha < 1$) or 1 (if $\alpha = 1$), as $n \to \infty$. This will imply (7.10) and (7.11). Considering expansions with respect to $h^{1/2}$ of ψ_P, ψ_Q, and φ_P near z_1, it is readily seen that

$$ie^{-2\varphi_P(z)}\sqrt{\frac{4\psi_P^2(z)+1}{4\psi_Q^2(z)+1}} = 1 + \mathcal{O}(h^{1/2}), \quad h \to 0,$$

while considering expansions with respect to $h^{1/2}$ of $\widehat{s}_n(1/\psi_P(z))$, $\widetilde{s}_n(-4\psi_Q(z))$ and $s_n(z)$ near z_1, it can be checked that

$$\frac{\widehat{s}_n(1/\psi_P(z))\widetilde{s}_n(-4\psi_Q(z))}{s_n(z)} = 1 + \mathcal{O}(h^{1/2}), \quad h \to 0,$$

where *the last $\mathcal{O}$ -term depends on n*. In view of (7.15) and the first equality in (7.16), we finally get that, as $n \to \infty$, the expression (7.14) is of order $\mathcal{O}(\log n)$ if $\alpha < 1$ and $\mathcal{O}(1)$ if $\alpha = 1$, uniformly in $|z-z_1| < \delta_n$, which finishes the proof of the lemma.

Proof of Corollary 2.14. The proof follows that of Corollary 2.12 of [27]. The behavior of the extreme zeros of P_n near z_1 is a consequence of the asymptotic formula (2.39). Indeed, we consider the function

$$F_n(t) = \frac{(-2)^{n+1}e^{-(n+1)(g_P(z)+\varphi_P(z))}}{\sqrt{\pi}n^{1/6}h_1(z)}P_n(z), \qquad \text{where } z = z_1 + tn^{-2/3}.$$

Then F_n has zeros $t_{\nu,n} = (z_{\nu,n}^P - z_1)n^{2/3}$, $\nu = 1, 2, \ldots$, and these zeros are ordered by increasing absolute value. Because $n^{-2/3} = o(\log^{-2} n)$, n large, the asymptotic formula

(2.39) applies, showing that

$$F_n(t) = \mathrm{Ai}\left((n+1)^{2/3} f_1(z_1 + tn^{-2/3})\right)\left(1 + \mathcal{O}\left(\frac{1}{n^\alpha}\right)\right) + n^{-1/3}\frac{h_2(z_1+tn^{-2/3})}{h_1(z_1+tn^{-2/3})}\mathrm{Ai}'\left((n+1)^{2/3} f_1(z_1+tn^{-2/3})\right) \times\left(1+\mathcal{O}\left(\frac{1}{n^\alpha}\right)\right). \tag{7.17}$$

Since f_1 is an analytic function with a simple zero at z_1 and $f_1'(z_1) = c_1$ (see (2.35), (2.36)), we get by expanding the function f_1 near z_1, and making use of (7.10) and (7.11), that

$$F_n(t) = \mathrm{Ai}\left(tc_1 + \mathcal{O}\left(n^{-2/3}\right)\right) + n^{-1/3}\frac{h_2(z_1+tn^{-2/3})}{h_1(z_1+tn^{-2/3})}\mathrm{Ai}'\left(tc_1+\mathcal{O}\left(n^{-2/3}\right)\right) + \mathcal{O}\left(\frac{1}{n^\alpha}\right). \tag{7.18}$$

Expanding the Airy function Ai near tc_1, and observing, thanks again to (7.10) and (7.11), that the second term of the sum in the right-hand side of (7.18) is of order $n^{-1/3}\log n$ if $0 < \alpha < 1$ and $n^{-1/3}$ if $\alpha = 1$, we get

$$F_n(t) = \mathrm{Ai}(tc_1) + \mathcal{O}\left(n^{2/3}\beta(n)\right), \tag{7.19}$$

where $\beta(n)$ has been defined in (2.42). The $\mathcal{O}$ -term holds uniformly on compact subsets of the complex t-plane. From Hurwitz' theorem it follows that for every fixed $v \in \mathbb{N}$, we have

$$\lim_{n\to\infty} t_{v,n} = -\frac{\iota_v}{c_1}.$$

Using the fact that $-\iota_v$ is a simple zero of the Airy function, we obtain from (7.19) that

$$t_{v,n} = -\frac{\iota_v}{c_1} + \mathcal{O}\left(n^{2/3}\beta(n)\right).$$

This proves (2.43), since $z^P_{v,n} = z_1 + t_{v,n}n^{-2/3}$.

The formulas (2.44) and (2.45) for the extreme zeros of Q_n and E_n near z_1 are obtained in a similar way from the asymptotics of Q_n and E_n near z_1, as given in Theorem 2.13.

7.3. Proofs of Theorems 2.7 and 2.2

Proof of Theorem 2.7. The limits for the counting measures ν_{P_n} and ν_{Q_n} follow from the strong asymptotic formulas (2.23) and (2.24), and the fact that, in view of (6.20), the family of functions $(s_n)_n$ satisfies

$$\frac{1}{n}\log s_n(z) \to 0, \qquad n \to 0,$$

locally uniformly in $\overline{\mathbb{C}} \setminus \{0\}$, in particular in a neighborhood of Γ_P and Γ_Q. The proof using the unicity theorem for logarithmic potentials (see e.g. [35, Theorem II.2.1]) essentially repeats the proof of [39, Theorem 2.1].

The proof for the limit of the measures ν_{E_n} is more difficult, since these measures have unbounded support and infinite mass. It can be adapted with minor changes from the proof of the similar statement about the measures ν_{E_n} in Theorem 2.5 of [27]. For the sake of completeness, we notice a difference when estimating an upper bound for the number $N_n(r)$ of roots of

$$f_n(z) = E_n(z)/\Omega_n(z) \tag{7.20}$$

with absolute value $\leqslant r$, see inequality (7.11) of [27]. We still start with the classical inequality in the theory of entire functions,

$$N_n(r) < \log \max_{|z|=er} |f_n(z)| - \log |f_n(0)|, \tag{7.21}$$

see [28, Section 2.5]. The second line in (2.25) implies that

$$f_n(0) = \frac{e^{-ng_P(0)}}{s_n(2)\sqrt{2}} \left(1 + \mathcal{O}\left(\frac{1}{n^\alpha}\right)\right). \tag{7.22}$$

Using $E_n(z) = P_n(z)e^{-nz} + Q_n(z)e^{nz}$ and the fact that the polynomials P_n and Q_n, of leading coefficients $s_n^{-2}(2)(1 + \mathcal{O}(1/n^\alpha))$ (see Corollary 2.11) and 1, respectively, have their zeros in a compact set, independently of n, we easily get that $|E_n(z)| \leqslant e^{2n|z|}$ for every $n \in \mathbb{N}$ and for every $|z| \geqslant R$ with R sufficiently large, say $R \geqslant R_0 \geqslant 1$. Then, from (7.21), (7.20) and (7.22) we see that there exists a constant $C > 0$ so that $N_n(r) < Cnr$ if $r > R_0$. The rest of the proof is similar to that of Theorem 2.5 of [27].

Proof of Theorem 2.2. Assertion (i) is a consequence of the strong asymptotics of the scaled rational interpolants in Theorem 2.10. In order to prove assertions (ii) and (iii), we use (2.11) and (4.4), and get the following expansions as $z \to 0$,

$$\psi_P(z) = -\frac{z}{4} + \mathcal{O}(z^3),$$
$$\psi_Q(z) = \frac{1}{z} + \mathcal{O}(z),$$
$$g_P(z) = (\log(2) - 1) + z - \frac{z^2}{4} + \mathcal{O}(z^3).$$

Making use of these expansions, the asymptotic formulas (2.23), (2.24) and (2.25), and plugging $z/2n$ instead of z, it is straightforward to check that (2.3) and (2.4) hold true.

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References

[1] A. Aptekarev, Sharp constants for rational approximations of analytic functions, Math. Sb. 193 (2002) 3–72 (translation in Sb. Math. 193 (2002) 1–72).

[2] A. Aptekarev, W. Van Assche, Scalar and matrix Riemann–Hilbert approach to the strong asymptotics of Padé approximants and complex orthogonal polynomials with varying weight, J. Approx. Theory 129 (2004) 129–166.

[3] J. Baik, P. Deift, K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, J. Amer. Math. Soc. 12 (1999) 1119–1178.

[4] J. Baik, P. Deift, K.T-R. McLaughlin, P. Miller, X. Zhou, Optimal tail estimates for directed last passage site percolation with geometric random variables, Adv. Theor. Math. Phys. 5 (2001) 1207–1250.

[5] J. Baik, T. Kriecherbauer, K.T.-R. McLaughlin, P.D. Miller, Uniform asymptotics for polynomials orthogonal with respect to a general class of discrete weights and universality results for associated ensembles: announcement of results, Internat. Math. Res. Notices 2003 (2003) 821–858.

[6] L. Baratchart, E.B. Saff, F. Wielonsky, Rational interpolation of the exponential function, Canad. J. Math. 47 (1995) 1121–1147.

[7] P. Bleher, A. Its, Semiclassical asymptotics of orthogonal polynomials, Riemann–Hilbert problem, and the universality in the matrix model, Ann. Math. 150 (1999) 185–266.

[8] P. Bleher, A. Its, Double scaling limit in the random matrix model: the Riemann–Hilbert approach, Comm. Pure Appl. Math. 56 (2003) 433–516.

[9] A.L. Cauchy, Sur la formule de Lagrange relative à l'interpolation, Analyse Algébrique, Paris, 1821.

[10] P. Deift, Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach, Courant Lecture Notes 3, New York University, 1999; American Mathematical Society, Providence RI, 2000.

[11] P. Deift, A. Its, X. Zhou, A Riemann–Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics, Ann. Math. 146 (1997) 149–235.

[12] P. Deift, T. Kriecherbauer, K.T-R. McLaughlin, S. Venakides, X. Zhou, Uniform asymptotics of polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, Commun. Pure Appl. Math. 52 (1999) 1335–1425.

[13] P. Deift, T. Kriecherbauer, K.T-R. McLaughlin, S. Venakides, X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, Commun. Pure Appl. Math. 52 (1999) 1491–1552.

[14] P. Deift, K.T-R. McLaughlin, A continuum limit of the Toda lattice, Mem. of Amer. Math. Soc. 131 (624), Providence RI, 1998.

[15] P. Deift, X. Zhou, A steepest descent method for oscillatory Riemann–Hilbert problems: asymptotics for the MKdV equation, Ann. Math. 137 (1993) 295–368.

[16] K.A. Driver, N.M. Temme, Asymptotics and zero distribution of Padé polynomials associated with the exponential function, J. Comput. Appl. Math. 89 (1998) 97–114.

[18] A.A. Gonchar, G. Lopez, On Markov's theorem for multipoint Padé approximants, Math. Sb. 34 (1978) 449–459.

[19] C. Hermite, Sur la fonction exponentielle, C. R. Acad. Sci. Paris 77 (1873) 18–24, 74–79, 226–233.

[20] J.C. Jacobi, Über die darstellung einer Reihe gegebener Werte durch eine gebrochene rationale Funktion, Crelle's J. Reine u. Angew. Math. 30 (1846) 127–156.

[21] T. Kriecherbauer, K. T-R McLaughlin, Strong asymptotics of polynomials orthogonal with respect to Freud weights, Internat. Math. Res. Notices 1999 (1999) 299–333.

[22] A.B.J. Kuijlaars, Riemann–Hilbert analysis for orthogonal polynomials, in: E. Koelink, W. Van Assche (Eds.), Orthogonal Polynomials and Special Functions, Lecture Notes in Mathematics, vol. 1817, Springer, Berlin, 2003, pp. 167–210.

[23] A.B.J. Kuijlaars, K. T-R McLaughlin, Riemann–Hilbert analysis for Laguerre polynomials with large negative parameter, Comput. Meth. Function Theory 1 (2001) 205–233.

[24] A.B.J. Kuijlaars, K. T-R McLaughlin, Asymptotic zero behavior of Laguerre polynomial with negative parameter, Constr. Approx. 20 (2004) 497–523.

[25] A.B.J. Kuijlaars, K. T-R McLaughlin, W. Van Assche, M. Vanlessen, The Riemann–Hilbert approach to strong asymptotics for orthogonal polynomials on $[-1, 1]$, Adv. Math. 188 (2004) 337–398.

[26] A.B.J. Kuijlaars, H. Stahl, W. Van Assche, F. Wielonsky, Asymptotique des approximants de Hermite-Padé quadratiques de la fonction exponentielle et problèmes de Riemann-Hilbert, C.R. Acad. Sci. Paris, Ser. I 336 (2003) 893–896.
[27] A.B.J. Kuijlaars, W. Van Assche, F. Wielonsky, Quadratic Hermite–Padé approximation to the exponential function: a Riemann–Hilbert approach, Constr. Approx., to appear.
[28] B.Y. Levin, Lectures on entire functions, Translations of Mathematical Monographs, vol. 150, American Mathematical Society, Providence RI, 1996.
[29] F. Olver, Asymptotics and Special Functions, Academic Press, San Diego, 1974.
[30] H. Padé, Sur la représentation approchée d'une fonction par des fractions rationelles, Ann. Sci. Ecole Norm. Sup 9 (3 Supplément) (1892) 3–93.
[31] H. Padé, Mémoire sur les développements en fractions continues de la fonction exponentielle pouvant servir d'introduction à la théorie des fractions continues algébriques, Ann. Sci. Ecole Norm. Sup. 16 (3) (1899) 395–426.
[32] H. Padé, Oeuvres, C. Brezinski, (Eds.), Librairie Scientifique et Technique A, Blanchard, Paris, 1984.
[33] O. Perron, Die Lehre von den Kettenbrüchen, 3rd ed., vol. 2, B.G. Teubner, Stuttgart, 1957.
[34] C. Pommerenke, Univalent Functions, Vandenhoeck & Ruprecht, Göttingen, 1975.
[35] E.B. Saff, V. Totik, Logarithmic Potentials with External Fields, Springer, Berlin, New York, 1997.
[36] E.B. Saff, R.S. Varga, On the zeros and poles of Padé approximants to e^z, Numer. Math. 25 (1975) 1–14.
[37] E.B. Saff, R.S. Varga, On the zeros and poles of Padé approximants to e^z. II., in: E.B. Saff, R.S. Varga (Eds.), Padé and Rational Approximations: Theory and Applications, Academic Press, New York, 1977, pp. 195–213.
[38] E.B. Saff, R.S. Varga, On the zeros and poles of Padé approximants to e^z. III., Numer. Math. 30 (1978) 241–266.
[39] H. Stahl, Quadratic Hermite–Padé polynomials associated with the exponential function, J. Approx. Theory 125 (2003) 238–294.
[40] H. Stahl, Asymptotic distributions of zeros of quadratic Hermite–Padé polynomials associated with the exponential function, manuscript, 2003.
[41] K. Strebel, Quadratic Differentials, Springer, Berlin, 1984.
[42] G. Szegő, Über eine Eigenschaft der Exponentialreihe, Sitzungsber. Berl. Math. Ges. 23 (1924) 50–64.
[43] R. Tijdeman, On the number of zeros of general exponential polynomials, Indag. Math. 33 (1971) 1–7.
[44] R.S. Varga, A.J. Carpenter, Asymptotics for the zeros and poles of normalized Padé approximants to e^z, Numer. Math. 68 (1994) 169–185.
[45] F. Wielonsky, On rational approximation to the exponential function with complex conjugate interpolation points, J. Approx. Theory 111 (2001) 344–368.